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ON LOCALLY COMPLEX ALGEBRAS AND LOW-DIMENSIONAL CAYLEY-DICKSON ALGEBRAS

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Abstract. The paper begins with short proofs of classical theorems by Frobenius and (resp.) Zorn on associative and (resp.) alternative real division algebras. These theorems characterize the first three (resp. four) Cayley-Dickson algebras. Then we introduce and study the class of real unital nonassociative algebras in which the subalgebra generated by any nonscalar element is isomorphic to $\mathbb{C}$. We call them *locally complex algebras*. In particular, we describe all such algebras that have dimension at most 4. Our main motivation, however, for introducing locally complex algebras is that this concept makes it possible for us to extend Frobenius' and Zorn's theorems in a way that it also involves the fifth Cayley-Dickson algebra, the sedenions.

1. Introduction

The real number field $\mathbb{R}$, the complex number field $\mathbb{C}$, and the division agebra of real quaternions $\mathbb{H}$ are classical examples of associative real division algebras. In 1878 Frobenius [10] proved that in the finite dimensional context they are also the only examples. Assuming alternativity instead of associativity, there is another example: $\mathbb{O}$, the division algebra of octonions. It turns out that this is the only additional example. This result is attributed to Zorn [21].

In Section 3 we give short and self-contained proofs of these classical theorems by Frobenius and Zorn. Both proofs are based on the same idea. In fact, the proof of Zorn's theorem is a continuation of the proof of Frobenius' theorem. The proofs are constructive, it appears like $\mathbb{H}$ and $\mathbb{O}$ are met "unintentionally".

Our proofs of Frobenius' and Zorn's theorems were discovered by accident, when examining the class of real unital algebras with the following property: the subalgebra generated by any element different from a scalar multiple of 1 is isomorphic to $\mathbb{C}$. These algebras, which we call *locally complex*, will be first considered in Section 4. In particular, we will classify all locally complex algebras of dimension at most 4.

Unlike real division algebras which exist only in dimensions $1, 2, 4$, and 8 [3, 13], locally complex algebras exist in abundance in any dimension. However, among alternative (and hence also associative) finite dimensional real algebras, the concepts of division algebras and locally complex algebras coincide. Frobenius' and Zorn's theorems can be therefore equivalently stated so that one replaces "division" by "locally complex" in the formulation. This observation paves the way for continuing in the direction of these two theorems.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the first four (real) algebras formed in the Cayley-Dickson process. The next one is the 16-dimensional algebra $\mathbb{S}$ of (real) *sedenions*. It is the first algebra in this process that is neither a division nor an alternative algebra. Although it is therefore somewhat less attractive than its famous predecessors, $\mathbb{S}$ has recently gained a considerable attention. Over the last years it was considered in several papers by algebraists as well as by mathematical physicists [1, 2, 4, 5, 6, 12, 14, 16]. To the best of our knowledge, however, there

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are no results that characterize $\mathbb{S}$ through its abstract algebraic properties. Moreover, one might get an impression when looking at some of these papers that such characterizations are not really expected (for example, see the introduction in [2]). One of the goals of this paper is to show that actually they can be established.

In Section 5 we consider locally complex algebras that are simultaneously superalgebras with the property that all their homogeneous elements satisfy the alternativity conditions (see (1) below). Our main result says that besides the obvious examples, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$, there are exactly two more algebras having these properties, one in dimension 8 and another one in dimension 16. As corollaries we get three characterizations of $\mathbb{S}$: the first one is based on the existence of special elements satisfying a version of the alternativity condition, the second one is based on the properties of zero divisors, and the third one is based on the structure of subalgebras.

Let us remark that among the papers listed above, the one by Calderon and Martin [5] is philosophically the closest one to our paper since it also considers superalgebras. However, the two papers do not seem to have any overlap. On the other hand, in our final results on sedenions we were influenced by the papers [2, 6, 16].

2. Preliminaries

The purpose of this section is to recall some definitions and elementary properties of the notions needed in subsequent sections.

Let A be a nonassociative algebra over a field. In this paper we will be actually interested only in the case where this field is $\mathbb{R}$, although some parts, like the following definitions and comments, make sense in a more general setting. Recall that A is said to be a *division algebra* if for every nonzero $a \in A$, $x \mapsto ax$ and $x \mapsto xa$ are bijective maps from A onto A. If A is finite dimensional, then this is clearly equivalent to the condition that A has no zero divisors. If A is associative, then it is a division algebra if and only if it is unital (i.e., it has a unity 1) and every nonzero element in A has a multiplicative inverse. For general algebras this is not true.

The real *Cayley-Dickson* algebras $\mathbb{A}_n$, $n \geq 0$, are (nonassociative) real algebras with involution $*$, defined recursively as follows: $\mathbb{A}_0 = \mathbb{R}$ with trivial involution $a^* = a$, and $\mathbb{A}_n$ is the vector space $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ endowed with multiplication and involution defined by

$$(a,b)(c,d) = (ac - d^*b, da + bc^*),$$

$$(a,b)^* = (a^*, -b).$$

It is easy to see that $\mathbb{A}_n$ is unital (in fact, the unity of $\mathbb{A}_n$ is $(1,0)$ where 1 is the unity of $\mathbb{A}_{n-1}$), $x + x^*$ and $xx^* = x^*x$ are scalar multiplies of 1 for every $x \in \mathbb{A}_n$, and $\dim \mathbb{A}_n = 2^n$. Next, it is clear that $\mathbb{A}_1 = \mathbb{C}$, and one easily notices that $\mathbb{A}_2 = \mathbb{H}$, the *quaternions*. The next algebra in this process is $\mathbb{A}_3 = \mathbb{O}$, the *octonions*. For an excellent survey on octonions we refer the reader to [1]. Let us record here just a few basic properties of $\mathbb{O}$. First of all, $\mathbb{O}$ is an 8-dimensional division algebra. Denoting its basis by $\{1, e_1, \ldots, e_7\}$, the multiplication in $\mathbb{O}$ is determined by the following table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Note that the linear span of $1, e_1, e_2, e_3$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

It is well known that $\mathbb{O}$ is a division algebra which is not associative. However, it is "almost" associative - namely, it is alternative. Recall that an algebra A is said to be *alternative* if

$$x^2y = x(xy) \quad \text{and} \quad yx^2 = (yx)x \tag{1}$$

holds for all $x, y \in A$. Incidentally, Artin's theorem says that this is equivalent to the condition that any two elements generate an associative subalgebra [20, p. 36]. We shall need the identities from (1) in their linearized forms:

$$(xz + zx)y = x(zy) + z(xy), \ y(xz + zx) = (yx)z + (yz)x. \tag{2}$$

Let us also record the so-called middle Moufang identity which, as one easily checks (see, e.g., [20, p. 35]), holds in every alternative algebra:

$$(xy)(zx) = x(yz)x. \tag{3}$$

The next algebra obtained by the Cayley-Dickson process is the 16-dimensional algebra $\mathbb{A}_4 = \mathbb{S}$, the *sedenions*. Let $\{1, e_1, \ldots, e_{15}\}$ be a basis of $\mathbb{S}$. This is the multiplication table for $\mathbb{S}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

The sedenions have zero divisors and they are not an alternative algebra. Anyhow, we shall see that they are close enough to alternative division algebras, so that these approximate properties are "almost" characteristic for $\mathbb{S}$. Let us recall the definition of another notion needed for dealing with these properties.

An algebra A is said to be a *superalgebra* if it is $\mathbb{Z}_2$-graded, i.e., there exist linear subspaces A_i, $i \in \mathbb{Z}_2$, such that $A = A_0 \oplus A_1$ and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. We call A_0 an *even* and A_1 an *odd* part of A. Elements in $A_0 \cup A_1$ are said to be *homogeneous*. Note that if A is unital, then $1 \in A_0$.

Cayley-Dickson algebras possess a natural superalgebra structure. Indeed, $A = \mathbb{A}_n$ becomes a superalgebra by defining $A_0 = \mathbb{A}_{n-1} \times 0$ and $A_1 = 0 \times \mathbb{A}_{n-1}$. This simple observation is the concept behind the contents of Section 5.

The algebras $\mathbb{A}_n$, $n \geq 4$, are not alternative, but at least they have certain nonscalar elements that share many properties with elements in alternative algebras: these are scalar multiples of the element $e = (0, 1)$, where 1 is of course the unity of $\mathbb{A}_{n-1}$ (see e.g. [2, Section 5]). Let us point out only one property that is sufficient for our purposes: e satisfies $x^2e = x(xe)$ for all $x \in \mathbb{A}_n$. This can be easily verified. Moreover, this property is "almost" characteristic for e: only elements in the linear span of 1 and e satisfy this identity for every x [8, Lemma 1.2] (the authors are thankful to Alberto Elduque for drawing their attention to this result). Now, let us call an element a in an arbitrary nonassociative algebra A an *alter-scalar* if a is not a scalar and satisfies $x^2a = x(xa)$ holds for all $x \in A$. (A similar, but not exactly the same notion of a strongly alternative element was defined in [17]. There is also a standard notion of an alternative element defined through the condition $a^2x = a(ax)$ for every x, but this is too weak for our goals). What is important for us is that $\mathbb{S}$ contains alter-scalars. With respect to the notation introduced above, these are nonzero scalar multiples of e_8. Thus, the standard basis of $\mathbb{S}$ has an element that is in some sense "better" than the others. This does not seem to be the case with the preceding Cayley-Dickson algebras.

Next we recall that an algebra A is said to be *quadratic* if it is unital and the elements $1, x, x^2$ are linearly dependent for every $x \in A$. Thus, for every $x \in A$ there exist $t(x), n(x) \in \mathbb{R}$ such that $x^2 - t(x)x + n(x) = 0$. Obviously, $t(x)$ and $n(x)$ are uniquely determined if $x \notin \mathbb{R}$. Setting $t(\lambda) = 2\lambda$ and $n(\lambda) = \lambda^2$ for $\lambda \in \mathbb{R}$, we can then consider t and n as maps from A into $\mathbb{R}$ (the reason for this definition is that in this way t becomes a linear functional, but we shall not need this). We call $t(x)$ and $n(x)$ the *trace* and the *norm* of x, respectively. For some elementary properties of quadratic algebras, a characterization of quadratic alternative algebras, and further references we refer to [9].

From $x^2-(x+x^*)x+x^*x = 0$ we see that all algebras $\mathbb{A}_n$ are quadratic. Further, every real division algebra A that is algebraic and power-associative (this means that every subalgebra generated by one element is associative) is automatically quadratic. Indeed, if $x \in A$ then there exists a nonzero polynomial $f(X) \in \mathbb{R}[X]$ such that $f(x) = 0$. Writing $f(X)$ as the product of linear and quadratic polynomials in $\mathbb{R}[X]$ it follows that $p(x) = 0$ for some $p(X) \in \mathbb{R}[X]$ of degree 1 or 2. In particular, algebraic alternative (and hence associative) real division algebras are quadratic.

Finally, if A is a real unital algebra, i.e., an algebra over $\mathbb{R}$ with unity 1, then we shall follow a standard convention and identify $\mathbb{R}$ with $\mathbb{R}1$; thus we shall write λ for $\lambda 1$, where $\lambda \in \mathbb{R}$.

3. Frobenius' and Zorn's theorems

Our first lemma is well known. It describes one of the basic properties of quadratic algebras. We give the proof for the sake of completness.

Lemma 3.1. *Let A be a quadratic real algebra. Then $U = \{u \in A\backslash\mathbb{R} \mid u^2 \in \mathbb{R}\}\cup\{0\}$ is a linear subspace of A, $uv + vu \in \mathbb{R}$ for all $u, v \in U$, and $A = \mathbb{R} \oplus U$.*

Proof. Obviously, U is closed under scalar multiplication. We have to show that $u, v \in U$ implies $u + v \in U$. If $u, v, 1$ are linearly dependent, then one easily notices that already u and v are dependent, and the result follows. Thus, let $u, v, 1$ be independent. We have $(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2 \in \mathbb{R}$. On the other hand, as A is quadratic there exist $\lambda, \mu \in \mathbb{R}$ such that $(u + v)^2 - \lambda(u + v) \in \mathbb{R}$ and $(u-v)^2-\mu(u-v) \in \mathbb{R}$, and hence $\lambda(u+v)+\mu(u-v) \in \mathbb{R}$. However, the independence of $1, u, v$ implies $\lambda + \mu = \lambda - \mu = 0$, so that $\lambda = \mu = 0$. This proves that $u \pm v \in U$. Thus U is indeed a subspace of A. Accordingly, $uv+vu = (u+v)^2 - u^2 - v^2 \in \mathbb{R}$ for all $u, v \in U$. Finally, if $a \in A \setminus \mathbb{R}$, then $a^2 - \nu a \in \mathbb{R}$ for some $\nu \in \mathbb{R}$, and therefore $u = a - \frac{\nu}{2} \in U$; thus, $a = \frac{\nu}{2} + u \in \mathbb{R} \oplus U$. □

Remark 3.2. If A is additionally a division algebra, then every nonzero $u \in U$ can be written as $u = \alpha v$ with $\alpha \in \mathbb{R}$ and $v^2 = -1$. Indeed, since $u^2 \in \mathbb{R}$ and since u^2 cannot be ≥ 0 – otherwise $(u - \alpha)(u + \alpha) = u^2 - \alpha^2$ would be 0 for some $\alpha \in \mathbb{R}$ – we have $u^2 = -\alpha^2$ with $0 \neq \alpha \in \mathbb{R}$. Thus, $v = \alpha^{-1}u$ is a desired element.

Note that by $\langle u, v\rangle = -\frac{1}{2}(uv + vu)$ one defines an inner product on U if A is a division algebra. The next lemma therefore deals with nothing but the Gram-Schmidt process. Nevertheless, we give the proof.

Lemma 3.3. *Let A be a quadratic real division algebra, and let U be as in Lemma 3.1. Suppose $e_1, \dots, e_k \in U$ are such that $e_i^2 = -1$ for all $i \leq k$ and $e_ie_j = -e_je_i$ for all $i, j \leq k$, $i \neq j$. If U is not equal to the linear span of $e_1, \dots, e_k$, then there exists $e_{k+1} \in U$ such that $e_{k+1}^2 = -1$ and $e_ie_{k+1} = -e_{k+1}e_i$ for all $i \leq k$.*

Proof. Pick $u \in U$ that is not contained in the linear span of $e_1, \dots, e_k$, and set $\alpha_i = \frac{1}{2}(ue_i + e_iu) \in \mathbb{R}$ (by Lemma 3.1). Note that $v = u + \alpha_1e_1 + \ldots + \alpha_ke_k$ satisfies

$e_iv = -ve_i$ for all $i \le k$. Let e_{k+1} be a scalar multiple of v such that $e_{k+1}^2 = -1$ (Remark 3.2). Then e_{k+1} has all desired properties. □

Theorem 3.4. (Frobenius' theorem) *An algebraic associative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. As pointed out at the end of Section 2, A is quadratic. We may assume that $n = \dim A \ge 2$. By Remark 3.2 we can fix $i \in A$ such that $i^2 = -1$. Thus, $A \cong \mathbb{C}$ if $n = 2$. Let $n > 2$. By Lemma 3.3 there is $j \in A$ such that $j^2 = -1$ and $ij = -ji$. Set $k = ij$. Now one immediately checks that $k^2 = -1$, $ki = j = -ik$, $jk = i = -kj$, and i, j, k are linearly independent. Therefore A contains a subalgebra isomorphic to $\mathbb{H}$. It remains to show that n is not > 4. If it was, then by Lemma 3.3 there would exist $e \in A$ such that $e \ne 0$, $ei = -ie$, $ej = -je$, and $ek = -ke$. However, from the first two identities we infer $eij = -iej = ije$; since $ij = k$, this contradicts the third identity. □

In standard graduate algebra textbooks one can find different proofs of Frobenius' theorem. In some of them the advanced theory is used, but there are also such that use only elementary tools, e.g., [11] and [15]. The proof in [11] is actually based on similar ideas than our proof, but it is considerably lengthier. The one in [15] (which is based on [18]) is different, and also short.

We believe that our proof, consisting of four simple steps (Lemma 3.1, Remark 3.2, Lemma 3.3, and the final proof), should be easily understandable to undergraduate students. Some of these steps, especially both lemmas, are of independent interest.

We now switch to the proof of Zorn's theorem. We need a simple lemma:

Lemma 3.5. *Let A be an alternative algebra, and let $e_1, \dots, e_k \in A$ be such that $e_ie_j \in \{e_1, \dots, e_k\}$ whenever $i \ne j$. If $w \in A$ is such that $e_iw = -we_i$ for every i, then $(e_ie_j)w = -e_i(e_jw)$ and $w(e_ie_j) = -(we_i)e_j$ whenever $i \ne j$.*

Proof. Just set $x = e_i$, $y = e_j$, and $z = w$ in (2), and the result follows. □

Theorem 3.6. (Zorn's theorem) *An algebraic alternative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. Since a subalgebra generated by two elements is associative, the first part of the proof of Theorem 3.4 remains unchanged in the present context. We may therefore assume that A contains a copy of $\mathbb{H}$ and that $n = \dim A > 4$. Let us just change the notation and write $e_1 = i$, $e_2 = j$, and $e_3 = k$. By Lemma 3.3 there exists $e_4 \in A$ such that $e_4^2 = -1$ and $e_4e_i = -e_ie_4$ for $i = 1, 2, 3$. Now define $e_5 = e_1e_4$, $e_6 = e_2e_4$, $e_7 = e_3e_4$. Using the alternativity and anticommutativity relations we see that

$$e_5^2 = e_6^2 = e_7^2 = -1,$$

$$e_1e_5 = -e_5e_1 = e_2e_6 = -e_6e_2 = e_3e_7 = -e_7e_3 = -e_4,$$

$$e_4e_5 = -e_5e_4 = e_1,\ e_4e_6 = -e_6e_4 = e_2,\ e_4e_7 = -e_7e_4 = e_3.$$

Further, using (3) we obtain

$$e_5e_6 = -e_6e_5 = -e_3,\ e_6e_7 = -e_7e_6 = -e_1,\ e_7e_5 = -e_5e_7 = -e_2.$$

Finally, use Lemma 3.5 with $k = 3$ and $w = e_4$, and note that the resulting identites yield the rest of the multiplication table.

It is easy to see that $1, e_1, \dots, e_7$ are linearly independent. Indeed, by taking squares we first see that $\sum_{i=1}^7 \lambda_ie_i$ cannot be a nonzero scalar; if $\sum_{i=1}^7 \lambda_ie_i = 0$, then after multiplying this relation with e_i we get $\lambda_i = 0$. Thus, we have showed that A contains $\mathbb{O}$.

It remains to show that $n = 8$. Suppose $n > 8$. Then, by Lemma 3.3, there exists $f \in A$ such that $f \neq 0$ and $fe_i = -e_i f$, $1 \leq i \leq 7$. Lemma 3.5 tells us that f also satisfies $(e_ie_j)f = -e_i(e_jf)$ and $f(e_ie_j) = -(fe_i)e_j$ for $i \neq j$. Accordingly,

$$e_1(e_2(e_4f)) = -e_1((e_2e_4)f) = -e_1(e_6f) = (e_1e_6)f = -e_7f. \tag{4}$$

Note that for $1 \leq i \leq 3$ we have

$$e_i(e_4f) = -(e_ie_4)f = f(e_ie_4) = -f(e_4e_i) = (fe_4)e_i = -(e_4f)e_i.$$

This makes it possible for us to apply Lemma 3.5 for $k = 3$ and $w = e_4f$. In particular this gives $(e_1e_2)(e_4f) = -e_1(e_2(e_4f))$. Consequently,

$$e_1(e_2(e_4f)) = -e_3(e_4f) = (e_3e_4)f = e_7f,$$

contradicting (4). □

Remark 3.7. From the first part of the proof we see that if an alternative (not necessarily a division) real algebra A contains a copy of $\mathbb{H}$ and $\dim A > 4$, then it also contains a copy of $\mathbb{O}$.

Classical versions of Frobenius' and Zorn's theorems deal with finite dimensional algebras rather than with (slightly more general) algebraic ones. Our method, however, yields these more general versions for free. But actually we shall need the more general version of Zorn's theorem in Section 5.

We cannot claim that any of the arguments given in this section is entirely original. After finding these proofs we have realized, when searching the literature, that many of these ideas appear in different texts. But to the best of our knowledge nobody has compiled these arguments in the same way that leads to short and direct proofs of theorems by Frobenius and Zorn. Therefore we hope and believe that this section is of some value.

4. Locally complex algebras

As already mentioned, we define a *locally complex algebra* as a real unital algebra A such that every $a \in A \setminus \mathbb{R}$ generates a subalgebra isomorphic to $\mathbb{C}$. A locally complex algebra A is obviously quadratic. We can therefore consider the trace $t(a)$ and the norm $n(a)$ of each $a \in A$.

Lemma 4.1. *The following conditions are equivalent for a real unital algebra A:*

- (i) *A is locally complex;*
- (ii) *every $0 \neq a \in A$ has a multiplicative inverse lying in $\mathbb{R}a + \mathbb{R}$;*
- (iii) *A is quadratic and A has no nontrivial idempotents or square-zero elements;*
- (iv) *A is quadratic and $n(a) > 0$ for every $0 \neq a \in A$.*

Moreover, if $2 \leq \dim A = n < \infty$, then (i)-(iv) *are equivalent to*

- (v) *A has a basis $\{1, e_1, \dots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_ie_j = -e_je_i$ for all $i \neq j$.*

Proof. It is easy to see that (i)$\Longrightarrow$ (ii) and (ii)$\Longrightarrow$ (iii). Suppose A is quadratic and $n(a) \leq 0$ for some $0 \neq a \in A$. Then $a \notin \mathbb{R}$. Therefore also $b = a - \frac{t(a)}{2} \notin \mathbb{R}$. Note that $b^2 \geq 0$. If $b^2 = 0$, then A has a nontrivial nilpotent. If $b^2 > 0$, i.e., $b^2 = \alpha^2$ for some $0 \neq \alpha \in \mathbb{R}$, then $e = \frac{1}{2}(1 - \alpha^{-1}b)$ is a nontrivial idempotent in A. Thus, (iii)$\Longrightarrow$ (iv). The proof of (iv)$\Longrightarrow$ (ii) is also straightforward. Therefore (ii)-(iv) are equivalent. Now assume (ii)-(iv) and pick $a \in A \setminus \mathbb{R}$. Then $b = a - \frac{t(a)}{2}$ satisfies $b^2 \in \mathbb{R}$. Just as in the argument above we see that b^2 cannot be ≥ 0. Hence $b^2 = -\alpha^2$ for some $\alpha \in \mathbb{R} \setminus \{0\}$, and so $i = \alpha^{-1}b$ satisfies $i^2 = -1$. This yields (i).

Finally, assume $2 \leq \dim A = n < \infty$. The implication (i)-(iv) $\Longrightarrow$ (v) follows from (the proof of) Lemma 3.3. Assuming (v) and writing $a \in A$ as $a = \lambda_0 + \sum_{i=1}^{n-1} \lambda_i e_i$, we see that $a^2 - t(a)a + n(a) = 0$ with $t(a) = 2\lambda_0$ and $n(a) = \sum_{i=0}^{n-1} \lambda_i^2$. Thus, (iv) holds. □

We can now list various examples of locally complex algebras.

Example 4.2. A quadratic real division algebra is locally complex.

Example 4.3. Let J_n be an n-dimensional real vector space, and let $\{1, e_1, \dots, e_{n-1}\}$ be its basis. Define a multiplication in J_n so that 1 is of course the unity, and the others are multiplied according to $e_i e_j = -\delta_{ij}$. Then J_n is a locally complex algebra and simultaneously a Jordan algebra. Another way of representing J_n is by identifying it with $\mathbb{R} \times \mathbb{R}^{n-1}$, and defining multiplication by $(\lambda, u)(\mu, v) = (\lambda\mu - \langle u, v\rangle, \lambda v + \mu u)$, where $\langle\, .\, ,\, .\, \rangle$ denotes the standard inner product on $\mathbb{R}^{n-1}$.

Example 4.4. A real unital algebra A is said to be *nicely normed* if there exists a linear map $* : A \to A$ such that $a^{**} = a$, $(ab)^* = b^*a^*$ for all $a, b \in A$, and $a+a^* \in \mathbb{R}$, $aa^* = a^*a > 0$ for all $0 \neq a \in A$ (cf. [1, p. 154]). These algebras form an important subclass of locally complex algebras. Namely, every element a in such an algebra A satisfies $a^2 - t(a)a + n(a) = 0$ with $t(a) = a + a^*$ and $n(a) = aa^*$, so that A is indeed locally complex. Note that $U = \{u \in A \setminus \mathbb{R} \,|\, u^2 \in \mathbb{R}\} \cup \{0\} = \{u \in A \,|\, u^* = -u\}$.

In particular, the Cayley-Dickson algebras $\mathbb{A}_n$ are nicely normed, and hence locally complex.

From Lemma 4.1 we can deduce the following characterization of finite dimensional nicely normed algebras.

Corollary 4.5. *let A be a real unital algebra. If $2 \leq \dim A = n < \infty$, then the following conditions are equivalent:*

(i) *A is nicely normed;*
(ii) *A has a basis $\{1, e_1, \dots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_i e_j = -e_j e_i \in \operatorname{span}\{e_1, \dots, e_{n-1}\}$ for all $i \neq j$.*

Proof. Assume (i). By Lemma 4.1 (v) A has a basis $\{1, e_1, \dots, e_{n-1}\}$ that has all desired properties except that we do not know yet that $e_i e_j \in \operatorname{span}\{e_1, \dots, e_{n-1}\}$. In view of the observation in Example 4.4 we have $\operatorname{span}\{e_1, \dots, e_{n-1}\} = U = \{u \in A \,|\, u^* = -u\}$. Therefore, if $i \neq j$, $(e_i e_j)^* = e_j^* e_i^* = e_j e_i = -e_i e_j$, and hence $e_i e_j \in U$. Conversely, if (ii) holds, then we can define $*$ according to $1^* = 1$ and $e_i^* = -e_i$, and one easily checks that this makes A a nicely normed algebra. ☐

If A is a *commutative* finite dimensional locally complex algebra, then the e_i's from (v) in Lemma 4.1 must satisfy $e_i e_j = 0$ if $i \neq j$. This can be interpreted as follows.

Corollary 4.6. *Let A be a locally complex algebra with $2 \leq \dim A = n < \infty$. Then A is commutative if and only if $A \cong J_n$.*

Let A be an alternative real algebra. If A is an algebraic division algebra, then it is quadratic, and hence locally complex. Conversely, if A is locally complex, then by Lemma 4.1 (ii) for every $0 \neq a \in A$ there exist $\lambda, \mu \in \mathbb{R}$ such that $a(\lambda a + \mu) = 1$. Since A is alternative it follows that for every $y \in A$ the equation $ax = y$ has the solution $x = (\lambda a + \mu)y$. Similarly one solves the equation $xa = y$. Therefore A is an algebraic division algebra. Accordingly, Frobenius' and Zorn's theorem can be equivalently stated as follows.

Theorem 4.7. (Frobenius' and Zorn's theorems) *An associative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. An alternative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

As already mentioned in the introduction, this version of Frobenius' and Zorn's theorems indicates the direction in which these theorems can be generalized. We shall deal with this in the next section.

In the rest of this section we will classify locally complex algebras up to dimesion 4. Clearly, $\mathbb{R}$ and $\mathbb{C}$ are, up to an isomorphism, the only locally complex algebras of dimension ≤ 2.

We fix some notation. The members of $\mathbb{R}\times\mathbb{R}^2$ will be denoted by $(\lambda,x)=(\lambda,x_1,x_2)$ and the members of $\mathbb{R}\times\mathbb{R}^3$ by $(\lambda,x)=(\lambda,x_1,x_2,x_3)$. For each (ordered) pair $x,y\in\mathbb{R}^2$ we denote by $|x\ y|$ the 2×2 determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. The symbol $x\times y$ stands for the usual vector product (cross product) of $x,y\in\mathbb{R}^3$, while (x,y,z) denotes the scalar triple product $(x,y,z)=\langle x\times y,z\rangle$, $x,y,z\in\mathbb{R}^3$.

Let t,s be nonnegative real numbers. We denote by $A_{t,s}$ the 3-dimensional algebra $A_{t,s}=\mathbb{R}\times\mathbb{R}^2$ with the multiplication given by

$$(\lambda,x)\,(\mu,y)=(\lambda\mu-\langle x,y\rangle+t|x\ y|,\lambda y+\mu x+s|x\ y|e_1),$$

where $e_1=(1,0)\in\mathbb{R}^2$. It follows from Lemma 4.1 (v) that $A_{t,s}$ is a locally complex algebra. We will show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t,s)\in[0,\infty)\times[0,\infty)$ and that $A_{t,s}$ and $A_{t',s'}$ are not isomorphic whenever $(t,s)\neq(t',s')$. In short, we have the following classification theorem for 3-dimensional locally complex algebras.

Theorem 4.8. *The map $(t,s)\mapsto A_{t,s}$, $t,s\geq 0$, induces a bijection between $[0,\infty)\times[0,\infty)$ and isomorphism classes of 3-dimensional locally complex algebras.*

Proof. We first show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t,s)\in[0,\infty)\times[0,\infty)$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R}\times\mathbb{R}^2$ with the multiplication given by

$$(\lambda,x)\,(\mu,y)=(\lambda\mu-\langle x,y\rangle,\lambda y+\mu x)+|x\ y|(t,z)$$

for some $(t,z)\in\mathbb{R}\times\mathbb{R}^2$. So, we may, and we will assume that A is this algebra. We have two possibilities; either $t\geq 0$, or $t<0$. Let us consider only the second one; the case when $t\geq 0$ can be handled in a similar, but simpler way. Set $s=\|z\|$. There exists an orthogonal 2×2 matrix Q such that $Qz=-se_1$ and $\det Q=-1$. Observe that $|Qx\ Qy|=(\det Q)|x\ y|=-|x\ y|$ and $\langle Qx,Qy\rangle=\langle x,y\rangle$, $x,y\in\mathbb{R}^2$. We claim that the map $\varphi: A\to A_{|t|,s}$ given by $\varphi(\lambda,x)=(\lambda,Qx)$, $(\lambda,x)\in\mathbb{R}\times\mathbb{R}^2$, is an isomorphism. Clearly, it is linear and bijective. Moreover, we have

$$\begin{aligned}\varphi((\lambda,x)\,(\mu,y))&=\varphi((\lambda\mu-\langle x,y\rangle+t|x\ y|,\lambda y+\mu x+|x\ y|z))\\&=(\lambda\mu-\langle x,y\rangle+t|x\ y|,\lambda Qy+\mu Qx-s|x\ y|e_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(\lambda,x)\,\varphi(\mu,y)&=(\lambda,Qx)\,(\mu,Qy)\\&=(\lambda\mu-\langle Qx,Qy\rangle+|t|\ |Qx\ Qy|,\lambda Qy+\mu Qx+s|Qx\ Qy|e_1)\\&=(\lambda\mu-\langle x,y\rangle+t|x\ y|,\lambda Qy+\mu Qx-s|x\ y|e_1).\end{aligned}$$

Hence, φ is an isomorphism. It remains to show that if $A_{t,s}$ and $A_{t',s'}$ are isomorphic for some $(t,s),(t',s')\in[0,\infty)\times[0,\infty)$, then $(t,s)=(t',s')$.

So, let $\varphi: A_{t,s}\to A_{t',s'}$ be an isomorphism. Then φ is linear and unital. In particular, $\varphi(\lambda,0)=(\lambda,0)$ for every $\lambda\in\mathbb{R}$. Furthermore, we have

$$\{(0,x)\in A_{t,s}\,|\,x\in\mathbb{R}^2\}=\{u\in A_{t,s}\,|\,u^2\in\mathbb{R}\ \text{ and }\ u\notin\mathbb{R}\}\cup\{0\}.$$

It follows that

$$\varphi(\lambda,x)=(\lambda,Qx)$$

for some linear map $Q:\mathbb{R}^2\to\mathbb{R}^2$. From

$$\begin{aligned}(\lambda^2-\|Qx\|^2,2\lambda Qx)&=(\lambda,Qx)^2=(\varphi(\lambda,x))^2\\&=\varphi((\lambda,x)^2)=\varphi(\lambda^2-\|x\|^2,2\lambda x)=(\lambda^2-\|x\|^2,2\lambda Qx)\end{aligned}$$

we get that $\|Qx\|^2=\|x\|^2$ for every $x\in\mathbb{R}^2$. Thus, Q is orthogonal. The equation

$$\varphi((\lambda,x)\,(\mu,y))=\varphi(\lambda,x)\,\varphi(\mu,y)$$

can be rewritten as

$$\begin{aligned}&(\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx + s|x\ y|Qe_1)\\ &= (\lambda\mu - \langle x, y\rangle + t'(\det Q)\,|x\ y|, \lambda Qy + \mu Qx + s'(\det Q)\,|x\ y|e_1).\end{aligned}$$

We conclude that $t = t' \det Q$ and $sQe_1 = s'(\det Q)e_1$. Applying the fact that $|\det Q| = 1$ and $\|Qe_1\| = \|e_1\| = 1$ we get $|t| = |t'|$ and $|s| = |s'|$. As all t, t', s, s' are nonnegative, we have $t = t'$ and $s = s'$, as desired. ☐

It follows directly from Corollary 4.5 that $A_{t,s}$ is nicely normed if and only if $t = 0$. So, the above statement shows that there is a natural bijection between $[0, \infty)$ and isomorphism classes of 3-dimensional nicely normed algebras.

The next result owes a lot to the paper [7] classifying 4-dimensional real quadratic division algebras. Our approach covers a more general class of real algebras. It is self-contained and completely elementary using just simple linear algebra tools.

We identify linear maps on $\mathbb{R}^3$ with 3×3 real matrices. Let M_3 denote the set of all 3×3 real matrices. For $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$ we write $(T, u) \sim (T', u')$ if and only if there exists an orthogonal 3×3 matrix Q such that $T' = (\det Q)QTQ^T$ and $u' = (\det Q)Qu$. It is clear that $\sim$ is an equivalence relation on $M_3 \times \mathbb{R}^3$. The set of equivalence classes will be denoted by $(M_3 \times \mathbb{R}^3)/\sim$.

For $T \in M_3$ and $u \in \mathbb{R}^3$ we denote by $A_{T,u}$ the 4-dimensional algebra $A_{T,u} = \mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y)).$$

As in the 3-dimensional case one can easily verify that $A_{T,u}$ is a locally complex algebra. We will show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$ and that $A_{T,u}$ and $A_{T',u'}$ are isomorphic if and only if $(T, u) \sim (T', u')$. In other words, we will prove the following.

Theorem 4.9. *The map $(T, u) \mapsto A_{T,u}$, $T \in M_3$, $u \in \mathbb{R}^3$, induces a bijection between $(M_3 \times \mathbb{R}^3)/\sim$ and isomorphism classes of 4-dimensional locally complex algebras.*

Proof. We will first show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle, \lambda y + \mu x) + S(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$$

for some linear map $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$. Observe that $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ can be decomposed into a direct sum of a linear functional on $\mathbb{R}^3$ and an endomorphism on $\mathbb{R}^3$. Recall that every linear functional on $\mathbb{R}^3$ can be represented in a unique way as an inner product with a fixed vector in $\mathbb{R}^3$. Finally, observe that the coordinates of the vector $(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$ are up to a permutation and a multiplication by ± 1 the coordinates of the vector product $x \times y$. Thus, A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y))$$

for some $u \in \mathbb{R}^3$ and some endomorphism T of $\mathbb{R}^3$. Hence, A is isomorphic to $A_{T,u}$, as desired.

Assume now that $A_{T,u}$ and $A_{T',u'}$ are isomorphic for some $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$. We have to show that $(T, u) \sim (T', u')$.

So, let $\varphi : A_{T,u} \to A_{T',u'}$ be an isomorphism. Exactly in the same way as in the 3-dimensional case we show that

$$\varphi(\lambda, x) = (\lambda, Qx)$$

for some orthogonal 3×3 matrix Q. The equation

$$\varphi((\lambda, x)\,(\mu, y)) = \varphi(\lambda, x)\,\varphi(\mu, y)$$

can be rewritten as

$$\begin{aligned}&(\lambda\mu - \langle x,y\rangle + (x,y,u), \lambda Qy + \mu Qx + QT(x\times y))\\ &= (\lambda\mu - \langle x,y\rangle + (Qx,Qy,u'), \lambda Qy + \mu Qx + T'(Qx\times Qy)).\end{aligned}$$

We conclude that

$$(x,y,u) = (Qx,Qy,u')$$

and

$$QT(x\times y) = T'(Qx\times Qy)$$

for all $x,y\in\mathbb{R}^3$. As Q is orthogonal we have $Q(x\times y) = (\det Q)(Qx\times Qy)$, and consequently,

$$(x,y,u) = (\det Q)\,(x,y,Q^Tu') \quad\text{and}\quad QT(x\times y) = (\det Q)\,T'Q(x\times y), \quad x,y\in\mathbb{R}^3.$$

It follows that $u' = (\det Q)Qu$ and $T' = (\det Q)QTQ^T$, as desired.

Finally, if $(T,u)\sim(T',u')$ for some $T,T'\in M_3$ and $u,u'\in\mathbb{R}^3$ then there exists an orthogonal 3×3 matrix Q such that $T' = (\det Q)QTQ^T$ and $u' = (\det Q)Qu$. It is then straightforward to check that the map $\varphi : A_{T,u}\to A_{T',u'}$ defined by $\varphi(\lambda,x) = (\lambda,Qx)$, $(\lambda,x)\in A_{T,u}$, is an isomorphism. □

It is rather easy to verify that $A_{T,u}$ is nicely normed if and only if $u=0$. We will next show that $A_{T,u}$ is a division algebra if and only if $\langle Tx,x\rangle\neq 0$ for each nonzero $x\in\mathbb{R}^3$ (that is, the quadratic form $q(x) = \langle Tx,x\rangle$ is either positive definite, or negative definite). Indeed, assume first that $A_{T,u}$ is not a division algebra. Then

$$(\lambda\mu - \langle x,y\rangle + (x,y,u), \lambda y + \mu x + T(x\times y)) = 0$$

for some nonzero $(\lambda,x),(\mu,y)\in A_{T,u}$. In particular,

$$T(x\times y) = -\lambda y - \mu x.$$

Set $z = x\times y$. We have $z\neq 0$, since otherwise x and y are linearly dependent and therefore

- either $\lambda = 0$ and then $\langle x,y\rangle = 0$ and $\mu x = 0$ which further yields that $(\lambda,x) = 0$ or $(\mu,y) = 0$, a contradiction; or
- $\mu = 0$ which yields a contradiction in exactly the same way; or
- $\lambda\neq 0$ and $\mu\neq 0$ and then $y = -\mu\lambda^{-1}x$ and $\lambda\mu = \langle x,y\rangle$ yield $0<\lambda^2 = -\langle x,x\rangle\leq 0$, a contradiction.

Hence, $z\neq 0$ and because z is orthogonal to both x and y we have $\langle Tz,z\rangle = 0$.

To prove the other direction we assume that there exists $z\in\mathbb{R}^3$ with $\|z\| = 1$ and $\langle Tz,z\rangle = 0$. Then $Tz = -tw$ for some real number t and some $w\in\mathbb{R}^3$ with $w\perp z$ and $\|w\| = 1$. There is a unique $v\in\mathbb{R}^3$ such that $z = w\times v$ and $v\perp w$. Set $s = -(w,v,u)$. Then $(0,w)$ and $(t, v - sw)$ are nonzero elements of $A_{T,u}$ whose product is equal to zero. Hence, $A_{T,u}$ is not a division algebra, as desired.

Following Dieterich's idea [7] we will now disscuss a geometric interpretation of the classification of 4-dimensional locally complex algebras. Let us start with a simple observation concerning 3×3 skew-symmetric matrices. If $x,y\in\mathbb{R}^3$ are any two vectors such that $x\times y = (c_1,c_2,c_3)$, then

$$R = \begin{bmatrix} 0 & c_3 & -c_2\\ -c_3 & 0 & c_1\\ c_2 & -c_1 & 0\end{bmatrix} = xy^T - yx^T,$$

where x and y are represented as 3×1 matrices. If Q is any orthogonal matrix, then $QRQ^T = (Qx)(Qy)^T - (Qy)(Qx)^T$. As $Qx\times Qy = (\det Q)\,Q(x\times y)$, we have

$$Q\begin{bmatrix} 0 & c_3 & -c_2\\ -c_3 & 0 & c_1\\ c_2 & -c_1 & 0\end{bmatrix}Q^T = \begin{bmatrix} 0 & d_3 & -d_2\\ -d_3 & 0 & d_1\\ d_2 & -d_1 & 0\end{bmatrix},$$

where

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = (\det Q)\, Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

If we choose $Q \in SO(3)$ such that

$$\begin{bmatrix} 0 \\ 0 \\ \sqrt{c_1^2 + c_2^2 + c_3^2} \end{bmatrix} = Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then

$$QRQ^T = \begin{bmatrix} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $d = \sqrt{c_1^2 + c_2^2 + c_3^2}$. In particular, $d = \|R\|$.

Any 3×3 matrix T can be uniquely decomposed into its symmetric and skew-symmetric part, $T = P + R$, $P = (1/2)(T + T^T)$, $R = (1/2)(T - T^T)$. If $T' = (\det Q)QTQ^T$ and $T' = P' + R'$ with P' symmetric and R' skew-symmetric, then $P' = (\det Q)QPQ^T$ and $R' = (\det Q)QRQ^T$. We will say that $A_{T,u}$ is of rank 3,2,1,0, respectively, if the symmetric part P of T is of rank 3,2,1,0, respectively. By the previous remark, two isomorphic algebras $A_{T,u}$ have the same rank.

Let us start with algebras $A_{T,u}$ of rank 3. We have two possibilities: either all eigenvalues of $P = T + T^T$ have the same sign, or P has both positive and negative eigenvalues. In the first case we will say that $A_{T,u}$ is an ellipsoid locally complex algebra of dimension 4, while in the second case we call $A_{T,u}$ a hyperboloid locally complex algebra of dimension 4. As we are interested in isomorphism classes we can use the fact that $A_{T,u}$ is isomorphic to $A_{-T,u}$ to restrict our attention to the case when all the eigenvalues of P are positive (the ellipsoid case) or to the case when two eigenvalues of P are positive and one is negative (the hyperboloid case). Once we have done this restriction two algebras $A_{T,u}$ and $A_{T',u'}$ of the above types are isomorphic if and only if $T' = QTQ^T$ and $u' = Qu$ for some $Q \in SO(3)$.

To consider isomorphism classes of hyperboloid locally complex algebras of dimension 4 (a 4-dimensional locally complex algebra is hyperboloid if it is isomorphic to some hyperboloid algebra $A_{T,u}$) we set $\tau = \{\delta \in \mathbb{R}^3 \,|\, \delta_1 \geq \delta_2 > 0 > \delta_3\}$ and $\kappa = \tau \times \mathbb{R}^3 \times \mathbb{R}^3$. The elements of κ will be called configurations. Each configuration consists of a hyperboloid $H_\delta = \{x \in \mathbb{R}^3 \,|\, \langle \Delta_\delta x, x\rangle = 1\}$ (a hyperboloid in principal axis form) and a pair of points. Here, Δ_δ is the diagonal matrix with the diagonal entries: $\delta_1, \delta_2, \delta_3$. The symmetry group of the hyperboloid H_δ is defined to be $G_\delta = \{Q \in SO(3) \,|\, Q\Delta_\delta Q^T = \Delta_\delta\}$ (the requirement that $\det Q = 1$ tells that we allow only symmetries that preserve the orientation). Note that this symmetry group consists of 4 elements whenever $\delta_1 > \delta_2$. Namely, in this case the symmetry group consists of the identity and all diagonal matrices with two eigenvalues -1 and one eigenvalue 1. The symmetry group is infinite if and only if the hyperboloid H_δ is circular, that is, $\delta_1 = \delta_2$. Two configurations (δ, u, c) and (δ', u', c') are said to be equivalent, $(\delta, u, c) \equiv (\delta', u', c')$, if and only if their hyperboloids coincide and their pairs of points lie in the same orbit under the operation of the symmetry group of the hyperboloid, that is, if and only if $\delta = \delta'$ and $(u', c') = (Qu, Qc)$ for some $Q \in G_\delta$. We denote by $\kappa/\equiv$ the set of equivalence classes of κ. We have a natural bijection between $\kappa/\equiv$ and the set of equivalence classes of hyperboloid locally complex algebras of dimension 4. Indeed, the bijection is induced by the map

$$(\delta, u, c) \mapsto A_{\Delta_\delta + R_c, u}$$

where

$$\Delta_\delta + R_c = \begin{bmatrix} \delta_1 & c_3 & -c_2 \\ -c_3 & \delta_2 & c_1 \\ c_2 & -c_1 & \delta_3 \end{bmatrix}.$$

Clearly, $A_{\Delta_\delta + R_c, u}$ is a hyperboloid locally complex algebra. We have to show that each hyperboloid algebra $A_{T,v}$ is isomorphic to some $A_{\Delta_\delta + R_c, u}$ and that $A_{\Delta_\delta + R_c, u}$ and $A_{\Delta_{\delta'} + R_{c'}, u'}$ are isomorphic if and only if $(\delta, u, c) \equiv (\delta', u', c')$. The second statement is trivial. To verify the first one we write $T = P + R$ with P symmetric with two positive eigenvalues and R skew-symmetric. Then there exists $Q \in SO(3)$ such that $QPQ^T = \Delta_\delta$ for some $\delta \in \tau$. We have $QRQ^T = R_c$ for some $c \in \mathbb{R}^3$. Set $u = Qv$ to complete the proof.

In a similar fashion we can consider isomorphism classes of ellipsoid locally complex algebras of dimension 4. Note that a locally complex algebra $A_{T,u}$ is a division algebra if and only if it is an ellipsoid algebra. As above we can consider configurations which consist of an ellipsoid in principal axis form and a pair of points. To each such configuration there corresponds a 4-dimensional real division algebra and this correspondence induces a bijection between the equivalence classes of configurations (the equivalence being defined via the symmetry group of the ellipsoid) and the isomorphism classes of 4-dimensional real quadratic division algebras. We omit the details that can be found in [7]. It is clear that locally complex algebras of rank 2 are either elliptic cylinder algebras or hyperbolic cylinder algebras. We leave the details to the reader. In the same way one can classify also isomorphism classes of locally complex algebras of rank 1. Let us conclude with the detailed disscussion on 4-dimensional locally complex algebras of rank 0. By e_3 we denote $e_3 = (0, 0, 1) \in \mathbb{R}^3$. We define an equivalence relation on the set $[0, \infty) \times \mathbb{R}^3$ as follows: $(d, u), (d', u') \in [0, \infty) \times \mathbb{R}^3$ are said to be equivalent, $(d, u) \equiv (d', u')$, if either

- $d = d' = 0$ and $\|u\| = \|u'\|$; or
- $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3 \rangle = \langle u', e_3 \rangle$.

Note that the equivalence class of $(d, u) \in [0, \infty) \times \mathbb{R}^3$ with $d > 0$ contains infinitely many elements if u and e_3 are linearly independent, and is a singleton when u is a scalar multiple of e_3. There is a natural bijection between the isomorphism classes of 4-dimensional locally complex algebras of rank 0 and the set $([0, \infty) \times \mathbb{R}^3)/\equiv$. The bijection is induced by the map from $[0, \infty) \times \mathbb{R}^3$ which maps the pair (d, u), $d \geq 0$, $u \in \mathbb{R}^3$, into $A_{T_d, u}$ with

$$T_d = \begin{bmatrix} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $A_{T_d, u}$ is a locally complex algebra of rank 0 and one can easily verify that each 4-dimensional locally complex algebra of rank 0 is isomorphic to some $A_{T_d, u}$. It remains to show that $A_{T_d, u}$ and $A_{T_{d'}, u'}$ are isomorphic if and only if $(d, u) \equiv (d', u')$. So, assume that $A_{T_d, u}$ and $A_{T_{d'}, u'}$ are isomorphic for some $(d, u), (d', u') \in [0, \infty) \times \mathbb{R}^3$. Then there exists an orthogonal matrix Q such that $T_{d'} = (\det Q) Q T_d Q^T$ and $u' = (\det Q) Q u$. In particular, $d' = \|T_{d'}\| = \|T_d\| = d$ and $\|u'\| = \|u\|$. If $d = 0$, then $d' = 0$, and hence, $(d, u) \equiv (d', u')$ in this special case. Therefore we may assume that $d = d' > 0$. From $T_{d'} = (\det Q) Q T_d Q^T$ we conclude that $Q e_3 = (\det Q) e_3$. Consequently,

$$\langle u', e_3 \rangle = \langle (\det Q) Q u, (\det Q) Q e_3 \rangle = \langle u, e_3 \rangle.$$

To prove the converse we assume that $(d, u) \equiv (d', u')$. We have one of the two possibilities and we will consider just the second one. So, assume that $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3 \rangle = \langle u', e_3 \rangle$. Then there exists an orthogonal matrix Q such

that $Qe_3 = e_3$ and $Qu = u'$. The orthogonal complement of e_3 and u is one-dimensional (if e_3 and u are linearly independent) or two-dimensional (if e_3 and u are linearly dependent). We have a freedom to choose the action of Q on the orthogonal complement of e_3 and u (of course, up to the requirement that Q is an orthogonal matrix). In particular, we can choose Q in such a way that $\det Q = 1$. It follows that $T_{d'} = QT_dQ^T$ and $u' = Qu$, as desired.

5. Super-alternative locally complex algebras

Let us call an algebra A a *super-alternative algebra* if it is a superalgebra, $A = A_0 \oplus A_1$, and the alternativity conditions (1) hold for all its homogeneous elements. Equivalently,

$$u^2x = u(ux),\ xu^2 = (xu)u \quad \text{for all } u \in A_i,\ i \in \mathbb{Z}_2,\ x \in A, \tag{5}$$

or, in the linearized form,

$$\begin{aligned} &(uv + vu)x = u(vx) + v(ux),\\ &x(uv + vu) = (xu)v + (xv)u \quad \text{for all } u, v \in A_i,\ i \in \mathbb{Z}_2\ ,\ x \in A. \end{aligned} \tag{6}$$

The notion of a super-alternative algebra should not be confused with the notion of an *alternative superalgebra*. The latter is defined through the alternativity of the Grassmann envelope of A. It turns out that nontrivial examples of alternative superalgebras exist only very exceptionally: prime alternative superalgebras of characteristic different from 2 and 3 are either associative or their odd part is zero [19]. As we shall see, super-alternative algebras are more easy to find.

Throughout this section A *will be a super-alternative locally complex algebra*. Our goal is to to classify all such algebras A. Obvious examples are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, as we can always take the trivial $\mathbb{Z}_2$-grading (the odd part is 0). Further, one can check by a straigtforward calculation that if $\mathbb{A}_{n-1}$ is an alternative algebra, then every $u \in (\mathbb{A}_{n-1} \times 0) \cup (0 \times \mathbb{A}_{n-1})$ satisfies (5) for every $x \in \mathbb{A}_n$. Therefore, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$ are super-alternative algebras with respect to the natural $\mathbb{Z}_2$-grading mentioned in Section 2. Of course, the important information for us in this context is that $\mathbb{S}$ is also a super-alternative locally complex algebra. As we shall see, besides $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and $\mathbb{S}$ only two more algebras must be added to the complete list of such algebras.

We continue by recording several simple but useful observations. First, the following special case of (6) will be often used:

(a) If $u, v \in A_i$, $i \in \mathbb{Z}_2$, are such that $uv + vu = 0$, then $u(vx) = -v(ux)$ and $(xu)v = -(xv)u$ for all $x \in A$.

If $v \in A_1$, then $v^2 \in A_0$; on the other hand, $v^2 = \lambda v + \mu$ for some $\lambda, \mu \in \mathbb{R}$. Since $v \notin A_0$, we must have $\lambda = 0$ and hence $v^2 = \mu \in \mathbb{R}$. Since A is locally complex, it follows that $\mu < 0$ if $v \neq 0$. Thus, we have

(b) If $0 \neq v \in A_1$, then there is $\alpha \in \mathbb{R}$ such that $(\alpha v)^2 = -1$.

Let $u \in A_0$ and $v \in A_1$ be such that $u^2 = v^2 = -1$. Using Lemma 3.1 we have $uv + vu \in \mathbb{R} \cap A_1 = 0$. Therefore $v(uv) = -v(vu) = -v^2u = u$. Next, $(uv)v = uv^2 = -u$. Similarly we see that $(uv)u = -u(uv) = v$. Finally, using (a) we get $(uv)(uv) = -(uv)(vu) = v((uv)u) = v^2 = -1$. We have proved:

(c) If $u \in A_0$ and $v \in A_1$ are such that $u^2 = v^2 = -1$, then $uv = -vu$, $v(uv) = -(uv)v = u$, $(uv)u = -u(uv) = v$, and $(uv)^2 = -1$.

Let u be a homogeneous element and suppose that $ux = 0$ for some $x \in A$. If $u \neq 0$, then by multiplying this identity from the left by $u - t(u)$ it follows from (5) that $n(u)x = 0$, and hence $x = 0$. Similarly, $xu = 0$ implies $x = 0$ if $u \neq 0$. Thus:

(d) Homogeneous elements are not zero divisors.

It is clear that our conditions on A imply that A_0 is a locally complex alternative algebra. Theorem 4.7 therefore tells us that A_0 is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. If $A_1 = 0$, then we get the desired conclusion that $A = A_0$ is one of the algebras from the expected list. Without loss of generality we may therefore assume that $A_1 \neq 0$. Given $0 \neq u \in A_1$, it follows from (d) that $x \mapsto ux$ is an injective linear map from A_0 into A_1; the same rule defines an injective linear map from A_1 into A_0. We may therefore conclude that

(e) $\dim A_0 = \dim A_1$.

In particular we now know that a super-alternative locally complex algebra must be finite dimensional. Moreover, its dimension can be only 1, 2, 4, 8, or 16.

We shall now consider separately each of the four possibilities concerning A_0.

Lemma 5.1. *If $A_0 \cong \mathbb{R}$, then $A \cong \mathbb{C}$.*

Proof. By (b) there is $i \in A_1$ with $i^2 = -1$, and hence $A \cong \mathbb{C}$ by (e). $\square$

Lemma 5.2. *If $A_0 \cong \mathbb{C}$, then $A \cong \mathbb{H}$.*

Proof. We have $A_0 = \mathbb{R} \oplus \mathbb{R}i$ with $i^2 = -1$. By (b) we may pick $j \in A_1$ such that $j^2 = -1$. Setting $k = ij \in A_1$ it follows from (c) that A contains a copy of $\mathbb{H}$. However, in view of (e) we actually have $A \cong \mathbb{H}$. $\square$

Let us now introduce another (an unexpected one for us) example of a super-alternative locally complex algebra. Let $\widetilde{\mathbb{O}}$ be the 8-dimensional algebra with basis $\{1, f_1, \ldots, f_7\}$ and multiplication table

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	f_7	$-f_6$
f_2	$-f_3$	-1	f_1	f_6	$-f_7$	$-f_4$	f_5
f_3	f_2	$-f_1$	-1	f_7	f_6	$-f_5$	$-f_4$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3
f_5	f_4	f_7	$-f_6$	$-f_1$	-1	f_3	$-f_2$
f_6	$-f_7$	f_4	f_5	$-f_2$	$-f_3$	-1	f_1
f_7	f_6	$-f_5$	f_4	$-f_3$	f_2	$-f_1$	-1

Lemma 5.3. *$\widetilde{\mathbb{O}}$ is a super-alternative locally complex algebra with zero divisors and without alter-scalar elements (and hence $\widetilde{\mathbb{O}} \ncong \mathbb{O}$).*

Proof. The fact that $\widetilde{\mathbb{O}}$ is locally complex follows from Lemma 4.1 (v). Let $\widetilde{\mathbb{O}}_0$ be the linear span of $1, f_1, f_2, f_3$, and let $\widetilde{\mathbb{O}}_1$ be the linear span of f_4, f_5, f_6, f_7. Then $\widetilde{\mathbb{O}}$ becomes a superalgebra with the even part $\widetilde{\mathbb{O}}_0 \cong \mathbb{H}$. From the way we shall arrive at $\widetilde{\mathbb{O}}$ in the next proof it is not really surprising that $\widetilde{\mathbb{O}}$ is super-alternative. But we used Mathematica for the actual checking that this is indeed true. Note that $(f_1 - f_4)(f_3 - f_6) = 0$, so that $\widetilde{\mathbb{O}}$ has zero divisors. Let $a \in A$ be such that $x^2 a = x(xa)$ for all $x \in \widetilde{\mathbb{O}}$. From $(f_i + f_j)^2 a = (f_i + f_j)((f_i + f_j)a)$, together with $f_i(f_i a) = f_j(f_j a) = -a$, it follows that $f_i(f_j a) + f_j(f_i a) = 0$ whenever $i \neq j$. Writing $a = \lambda_0 + \sum_{k=1}^{7} \lambda_k f_k$ we thus have

$$\sum_{k=1}^{7} \lambda_k \Big(f_i(f_j f_k) + f_j(f_i f_k) \Big) = 0 \quad \text{whenever } i \neq j. \tag{7}$$

Chosing $i = 1$ and $j = 4$ it follows that $\lambda_2 = \lambda_3 = \lambda_6 = \lambda_7 = 0$. Chosing, for example, $i = 2$ and $j = 7$ we further get $\lambda_1 = \lambda_4 = 0$, and chosing $i = 3$ and $j = 4$ finally leads to $\lambda_5 = 0$. Therefore $a = \lambda_0$ is a scalar. $\square$

Lemma 5.4. *If $A_0 \cong \mathbb{H}$, then $A \cong \mathbb{O}$ or $A \cong \widetilde{\mathbb{O}}$.*

Proof. Let $\{1, i, j, k\}$ be a basis of A_0 where these elements have the usual meaning. Pick $f \in A_1$ with $f^2 = -1$. Then f anticommutes with i, j, k by (c). It is clear that $\{f, if, jf, kf\}$ is a basis of A_1. We claim that all elements in this basis pairwise anticommute. It is easy to see that f anticommutes with each of if, jf, kf. Using (a) repeatedly we obtain $(if)(jf) = -(i(jf))f = (j(if))f = -(jf)(if)$. Other identities can be checked analogously.

Since $i(jf) \in A_1$, we have

$$i(jf) = \lambda_1 f + \lambda_2 if + \lambda_3 jf + \lambda_4 kf \tag{8}$$

for some $\lambda_i \in \mathbb{R}$. From (a) we infer that $(i(jf))f = -(if)(jf)$. Similarly, using (a) and (c) we get

$$f(i(jf)) = -f((jf)i) = (jf)(fi) = -(jf)(if) = (if)(jf).$$

The last two identities show that $i(jf)$ anticommutes with f. Consequently, anticommuting (8) with f it follows that $\lambda_1 = 0$. A similar arguing shows that $i(jf)$ anticommutes with both if and jf, which leads to $\lambda_2 = \lambda_3 = 0$. Note that (c) implies that the squares of both kf and $i(jf)$ are equal -1. But then $\lambda_4^2 = 1$, i.e., $\lambda_4 = 1$ or $\lambda_4 = -1$. If $\lambda_4 = 1$, i.e., $i(jf) = kf$, then we set $f_1 = i$, $f_2 = j$, $f_3 = k$, $f_4 = f$, $f_5 = if$, $f_6 = jf$, and $f_7 = kf$. Using the information we have, it is now just a matter of a routine calculation to verify that $A \cong \widetilde{\mathbb{O}}$. Since we know that $\mathbb{O}$ is a super-alternative locally complex algebra, the other possibility $\lambda_4 = -1$ can lead only to $A \cong \mathbb{O}$. $\square$

The 16-dimensional analogue of $\widetilde{\mathbb{O}}$ is the algebra which we denote by $\widetilde{\mathbb{S}}$ and define as follows: if $\{1, f_1, \ldots, f_{15}\}$ is its basis, then the multiplication table is

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	$-f_7$	f_6	f_9	$-f_8$	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}
f_2	$-f_3$	-1	f_1	f_6	f_7	$-f_4$	$-f_5$	f_{10}	f_{11}	$-f_8$	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$
f_3	f_2	$-f_1$	-1	f_7	$-f_6$	f_5	$-f_4$	f_{11}	$-f_{10}$	f_9	$-f_8$	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3	f_{12}	f_{13}	f_{14}	$-f_{15}$	$-f_8$	$-f_9$	$-f_{10}$	f_{11}
f_5	f_4	$-f_7$	f_6	$-f_1$	-1	$-f_3$	f_2	f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	$-f_8$	f_{11}	f_{10}
f_6	f_7	f_4	$-f_5$	$-f_2$	f_3	-1	$-f_1$	f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	$-f_8$	$-f_9$
f_7	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1	f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	$-f_8$
f_8	$-f_9$	$-f_{10}$	$-f_{11}$	$-f_{12}$	$-f_{13}$	$-f_{14}$	$-f_{15}$	-1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_9	f_8	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}	$-f_1$	-1	$-f_3$	f_2	$-f_5$	f_4	$-f_7$	f_6
f_{10}	f_{11}	f_8	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$	$-f_2$	f_3	-1	$-f_1$	$-f_6$	f_7	f_4	$-f_5$
f_{11}	$-f_{10}$	f_9	f_8	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$	$-f_3$	$-f_2$	f_1	-1	f_7	f_6	$-f_5$	$-f_4$
f_{12}	f_{13}	f_{14}	$-f_{15}$	f_8	$-f_9$	$-f_{10}$	f_{11}	$-f_4$	f_5	f_6	$-f_7$	-1	$-f_1$	$-f_2$	f_3
f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	f_8	f_{11}	f_{10}	$-f_5$	$-f_4$	$-f_7$	$-f_6$	f_1	-1	f_3	f_2
f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	f_8	$-f_9$	$-f_6$	f_7	$-f_4$	f_5	f_2	$-f_3$	-1	$-f_1$
f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	f_8	$-f_7$	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1

The proof of the next lemma is similar to that of Lemma 5.3. Therefore we omit details.

Lemma 5.5. *$\widetilde{\mathbb{S}}$ is a super-alternative locally complex algebra without alter-scalar elements (and hence $\widetilde{\mathbb{S}} \ncong \mathbb{S}$).*

The final lemma has a similar statement than Lemma 5.4, but the proof is somewhat more complicated. One of the problems that we have to face in this proof is that we do not have a complete freedom in the selection of an element playing the role of f from the proof of Lemma 5.4. While f was an arbitrary element in A_1 with square -1, now we shall have to find a special one.

Lemma 5.6. *If $A_0 \cong \mathbb{O}$, then $A \cong \mathbb{S}$ or $A \cong \widetilde{\mathbb{S}}$.*

Proof. Let $\{1, e_1, \ldots, e_7\}$ be a basis of A_0 whose multiplication table is given in Section 2. We begin with three claims needed for future reference.

CLAIM 1: Let $i, j \in \{1, 2, \ldots, 7\}$, $i \neq j$. If $p \in A_1$, then $q = p + (e_i e_j)(e_i(e_j p))$ satisfies $(e_i e_j)q = -e_i(e_j q)$.

Indeed, by (5) we have $(e_ie_j)q = (e_ie_j)p - e_i(e_jp)$, while using (a) and (5) we get

$$\begin{aligned} e_i(e_jq) &= e_i(e_jp) + e_i(e_j((e_ie_j)(e_i(e_jp)))) = e_i(e_jp) - e_i((e_ie_j)(e_j(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_i(e_j(e_i(e_jp)))) = e_i(e_jp) - (e_ie_j)(e_j(e_i(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_j(e_jp)) = e_i(e_jp) - (e_ie_j)p, \end{aligned}$$

so that $(e_ie_j)q = -e_i(e_jq)$.

Claim 2: Let $i, j, k \in \{1, 2, \dots, 7\}$ be such that e_i, e_j, e_ie_j, e_k are linearly independent, and let $s \in A_1$ be such that $(e_ie_j)s = -e_i(e_js)$. Then $t = s + (e_ie_k)(e_i(e_ks))$ also satisfies $(e_ie_j)t = -e_i(e_jt)$.

(Let us add that (a) implies $t = s + (e_ke_i)(e_k(e_is))$, and that $(e_ie_j)z = -e_i(e_jz)$ is equivalent to $(e_je_i)z = -e_j(e_iz)$; the order of indices is thus irrelevant.)

Indeed, by now already familiar arguing we have

$$\begin{aligned} (e_ie_j)t &= (e_ie_j)s + (e_ie_j)((e_ie_k)(e_i(e_ks))) = (e_ie_j)s - (e_ie_k)((e_ie_j)(e_i(e_ks))) \\ &= (e_ie_j)s + (e_ie_k)(e_i((e_ie_j)(e_ks))) = (e_ie_j)s - (e_ie_k)(e_i(e_k((e_ie_j)s))) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_i(e_k(e_i(e_js))))\big) = -\big(e_i(e_js) + (e_ie_k)(e_k(e_i(e_i(e_js))))\big) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_k(e_js))\big) = -\big(e_i(e_js) + e_i(e_i((e_ie_k)(e_k(e_js))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_i(e_k(e_js))))\big) = -\big(e_i(e_js) + e_i((e_ie_k)(e_i(e_j(e_ks))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_j(e_i(e_ks))))\big) = -\big(e_i(e_js) + e_i(e_j((e_ie_k)(e_i(e_ks))))\big) \\ &= -e_i(e_jt). \end{aligned}$$

Claim 3: Let $i, j, k \in \{1, 2, \dots, 7\}$, $i \neq j$, and let $\epsilon \in \mathbb{R}$ and $w \in A_1$ be such that $(e_ie_j)w = \epsilon e_i(e_jw)$. Set $u = e_kw$. If $k \in \{i, j\}$, then $(e_ie_j)u = \epsilon e_i(e_ju)$, and if $k \notin \{i, j\}$, then $(e_ie_j)u = -\epsilon e_i(e_ju)$.

If $k \in \{i, j\}$, then we may assume $k = j$ without loss of generality. We have

$$(e_ie_j)(u) = (e_ie_j)(e_jw) = -e_j((e_ie_j)w) = -\epsilon e_j(e_i(e_jw)) = \epsilon e_i(e_ju).$$

If $k \notin \{i, j\}$, then we have

$$\begin{aligned} (e_ie_j)(u) &= (e_ie_j)(e_kw) = -e_k((e_ie_j)w) \\ &= -\epsilon e_k(e_i(e_jw)) = \epsilon e_i(e_k(e_jw)) = -\epsilon e_i(e_ju). \end{aligned}$$

After establishing these auxiliary claims, we now begin the actual proof by picking a nonzero $u \in A_1$. As mentioned above, an arbitrary chosen u may not be the right choice, so we have to "remedy" it. Let $v' = u + (e_1e_2)(e_1(e_2u)) \in A_1$. By Claim 1, v' satisfies $(e_1e_2)v' = -e_1(e_2v')$. If $v' = 0$, then we have $(e_1e_2)u = e_1(e_2u)$. But then $v'' = e_3u$ satisfies $(e_1e_2)v'' = -e_1(e_2v'')$ by Claim 3. Thus, in any case there is a nonzero $v \in A_1$ such that

$$(e_1e_2)v = -e_1(e_2v).$$

Now consider $w' = v + (e_1e_4)(e_1(e_4v))$. By Claim 1 we have $(e_1e_4)w' = -e_1(e_4w')$, and by Claim 2 we have $(e_1e_2)w' = -e_1(e_2w')$. If $w' = 0$, then $(e_1e_4)v = e_1(e_4v)$. But then $w'' = e_2v$ satisfies $(e_1e_2)w'' = -e_1(e_2w'')$ and $(e_1e_4)w'' = -e_1(e_4w'')$. Thus, there exists a nonzero $w \in A_1$ satisfying

$$(e_1e_2)w = -e_1(e_2w), \ (e_1e_4)w = -e_1(e_4w).$$

We now repeat the same procedure with respect to e_2 and e_4. That is, we introduce $x' = w + (e_2e_4)(e_2(e_4w))$, and apply Claims 1 and 2 to conclude that $(e_1e_2)x' = -e_1(e_2x')$, $(e_1e_4)x' = -e_1(e_4x')$, and $(e_2e_4)x' = -e_2(e_4x')$. If $x' = 0$, then $(e_2e_4)w = e_2(e_4w)$, and therefore Claim 3 tells us that $(e_1e_2)x'' = -e_1(e_2x'')$, $(e_1e_4)x'' = -e_1(e_4x'')$, and $(e_2e_4)x'' = -e_2(e_4x'')$, where $x'' = e_1w$. In any case we have found a a nonzero $x \in A_1$ satisfying

$$(e_1e_2)x = -e_1(e_2x), \ (e_1e_4)x = -e_1(e_4x), \ (e_2e_4)x = -e_2(e_4x).$$

Considering $y' = x + (e_3e_4)(e_3(e_4x))$ we see from Claim 2 that $(e_1e_4)y' = -e_1(e_4y')$ and $(e_2e_4)y' = -e_2(e_4y')$, while apparently we cannot conclude that also $(e_1e_2)y' = -e_1(e_2y')$. However, multiplying $(e_1e_2)x = -e_1(e_2x)$ from the left by e_1 we get $e_1((e_1e_2)x) = e_2x$, which can be written as $e_1(e_3x) = -(e_1e_3)x$. Therefore Claim 2 yields $e_1(e_3y') = -(e_1e_3)y'$. Multiplying this from the left by e_1 we arrive at the desired identity $(e_1e_2)y' = -e_1(e_2y')$. Also, $(e_3e_4)y' = -e_3(e_4y')$ holds by Claim 1. We still have to deal with the case where $y' = 0$, i.e., $(e_3e_4)x = e_3(e_4x)$. The usual reasoning now does not work, since we do not have "enough room" to apply Claim 3. Thus, the final conclusion is that there exists a nonzero $y \in A_1$ such that

$$(e_1e_2)y = -e_1(e_2y),\ (e_1e_4)y = -e_1(e_4y),\ (e_2e_4)y = -e_2(e_4y),\ (e_3e_4)y = \pm e_3(e_4y).$$

In view of (b) we may assume without loss of generality that $y^2 = -1$. Let us first consider the case where $(e_3e_4)y = e_3(e_4y)$. We set $f_8 = y$ and $f_i = e_i$, $f_{i+8} = f_if_8$, $i = 1, \dots, 7$. By standard calculations one can now verify that $A \cong \widetilde{\mathbb{S}}$; checking all details is lengthy and tedious, but straigtforward. The other possibility where $(e_3e_4)y = -e_3(e_4y)$ of course leads to $A \cong \mathbb{S}$. □

All lemmas together yield our main result.

Theorem 5.7. *A super-alternative locally complex algebra is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\widetilde{\mathbb{O}}$, $\mathbb{S}$, *or* $\widetilde{\mathbb{S}}$.

Remark 5.8. In the course of the proof we did not use the assumption that (5) holds for all $u, x \in A_1$. Therefore we can replace the super-alternativity assumption by a slightly milder one.

This list reduces to Cayley-Dickson algebras under the additional assumption that there exist alter-scalar elements.

Corollary 5.9. *A super-alternative locally complex algebra containing alter-scalar elements is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, *or* $\mathbb{S}$.

Corollary 5.10. *A super-alternative locally complex algebra which contains alter-scalar elements, but is not alternative, is isomorphic to* $\mathbb{S}$.

Let A be an algebra, and let $x \in A$. The *annihilator* of x is the space $\mathrm{Ann}(x) = \{y \in A \,|\, xy = 0\}$. If $A = \mathbb{A}_n$ is a Cayley-Dickson algebra, then the dimension of $\mathrm{Ann}(x)$ is a multiple of 4 [2, 16]. Moreover, if $A = \mathbb{A}_4 = \mathbb{S}$, then the dimension of $\mathrm{Ann}(x)$ is exactly 4 for every zero divisor x in A [2, Section 12]. The algebras $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbb{S}}$ do not have this property. It is easy to check that $x = f_1 - f_4 \in \widetilde{\mathbb{O}}$ has the 2-dimensional annihilator spanned by $f_2 + f_7$ and $f_3 - f_6$. Further, the dimension of the annihilator of $x = f_3 + f_{12} \in \widetilde{\mathbb{S}}$ is 6; it is spanned by $f_1 + f_{14}$, $f_2 - f_{13}$, $f_4 + f_{11}$, $f_5 + f_{10}$, $f_6 - f_9$, and $f_7 - f_8$. Thus, we have

Corollary 5.11. *Let A be a super-alternative locally complex algebra which is not a division algebra. If the dimension of* $\mathrm{Ann}(x)$ *is* 4 *for every zero divisor in A, then* $A \cong \mathbb{S}$.

One can check that

$$1 \mapsto 1,\ e_1 \mapsto f_1,\ e_2 \mapsto f_2,\ e_3 \mapsto f_3,\ e_4 \mapsto f_{12},\ e_5 \mapsto -f_{13},\ e_6 \mapsto -f_{14},\ e_7 \mapsto -f_{15}$$

defines an embedding of $\widetilde{\mathbb{O}}$ into $\mathbb{S}$. Thus, both $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ can be viewed as subalgebras of $\mathbb{S}$. Chan and Đoković proved that $\mathbb{S}$ has 6-dimensional subalgebras, which, however, are not contained in 8-dimensional subalgebras of $\mathbb{S}$ [6, Corollary 3.6, Theorem 8.1]. Accordingly, $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ do not have 6-dimensional subalgebras. Further, $\mathbb{S}$ does not contain 5-dimensional subalgebras [6, Proposition 4.4]. This does not hold for $\widetilde{\mathbb{S}}$. For example, the linear span of 1, $f_1 + f_{14}$, $f_3 - f_{12}$, $f_6 - f_9$, and $f_7 - f_8$ is a 5-dimensional subalgebra of $\widetilde{\mathbb{S}}$. Combining all these we get our final corollary.

Corollary 5.12. *Let A be a super-alternative locally complex algebra. If A contains 6-dimensional subalgebras, but does not contain 5-dimensional subalgebras, then $A \cong \mathbb{S}$.*

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ON LOCALLY COMPLEX ALGEBRAS AND LOW-DIMENSIONAL CAYLEY-DICKSON ALGEBRAS

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Abstract. The paper begins with short proofs of classical theorems by Frobenius and (resp.) Zorn on associative and (resp.) alternative real division algebras. These theorems characterize the first three (resp. four) Cayley-Dickson algebras. Then we introduce and study the class of real unital nonassociative algebras in which the subalgebra generated by any nonscalar element is isomorphic to $\mathbb{C}$. We call them *locally complex algebras*. In particular, we describe all such algebras that have dimension at most 4. Our main motivation, however, for introducing locally complex algebras is that this concept makes it possible for us to extend Frobenius' and Zorn's theorems in a way that it also involves the fifth Cayley-Dickson algebra, the sedenions.

1. Introduction

The real number field $\mathbb{R}$, the complex number field $\mathbb{C}$, and the division agebra of real quaternions $\mathbb{H}$ are classical examples of associative real division algebras. In 1878 Frobenius [10] proved that in the finite dimensional context they are also the only examples. Assuming alternativity instead of associativity, there is another example: $\mathbb{O}$, the division algebra of octonions. It turns out that this is the only additional example. This result is attributed to Zorn [21].

In Section 3 we give short and self-contained proofs of these classical theorems by Frobenius and Zorn. Both proofs are based on the same idea. In fact, the proof of Zorn's theorem is a continuation of the proof of Frobenius' theorem. The proofs are constructive, it appears like $\mathbb{H}$ and $\mathbb{O}$ are met "unintentionally".

Our proofs of Frobenius' and Zorn's theorems were discovered by accident, when examining the class of real unital algebras with the following property: the subalgebra generated by any element different from a scalar multiple of 1 is isomorphic to $\mathbb{C}$. These algebras, which we call *locally complex*, will be first considered in Section 4. In particular, we will classify all locally complex algebras of dimension at most 4.

Unlike real division algebras which exist only in dimensions $1, 2, 4$, and 8 [3, 13], locally complex algebras exist in abundance in any dimension. However, among alternative (and hence also associative) finite dimensional real algebras, the concepts of division algebras and locally complex algebras coincide. Frobenius' and Zorn's theorems can be therefore equivalently stated so that one replaces "division" by "locally complex" in the formulation. This observation paves the way for continuing in the direction of these two theorems.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the first four (real) algebras formed in the Cayley-Dickson process. The next one is the 16-dimensional algebra $\mathbb{S}$ of (real) *sedenions*. It is the first algebra in this process that is neither a division nor an alternative algebra. Although it is therefore somewhat less attractive than its famous predecessors, $\mathbb{S}$ has recently gained a considerable attention. Over the last years it was considered in several papers by algebraists as well as by mathematical physicists [1, 2, 4, 5, 6, 12, 14, 16]. To the best of our knowledge, however, there

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are no results that characterize $\mathbb{S}$ through its abstract algebraic properties. Moreover, one might get an impression when looking at some of these papers that such characterizations are not really expected (for example, see the introduction in [2]). One of the goals of this paper is to show that actually they can be established.

In Section 5 we consider locally complex algebras that are simultaneously superalgebras with the property that all their homogeneous elements satisfy the alternativity conditions (see (1) below). Our main result says that besides the obvious examples, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$, there are exactly two more algebras having these properties, one in dimension 8 and another one in dimension 16. As corollaries we get three characterizations of $\mathbb{S}$: the first one is based on the existence of special elements satisfying a version of the alternativity condition, the second one is based on the properties of zero divisors, and the third one is based on the structure of subalgebras.

Let us remark that among the papers listed above, the one by Calderon and Martin [5] is philosophically the closest one to our paper since it also considers superalgebras. However, the two papers do not seem to have any overlap. On the other hand, in our final results on sedenions we were influenced by the papers [2, 6, 16].

2. Preliminaries

The purpose of this section is to recall some definitions and elementary properties of the notions needed in subsequent sections.

Let A be a nonassociative algebra over a field. In this paper we will be actually interested only in the case where this field is $\mathbb{R}$, although some parts, like the following definitions and comments, make sense in a more general setting. Recall that A is said to be a *division algebra* if for every nonzero $a \in A$, $x \mapsto ax$ and $x \mapsto xa$ are bijective maps from A onto A. If A is finite dimensional, then this is clearly equivalent to the condition that A has no zero divisors. If A is associative, then it is a division algebra if and only if it is unital (i.e., it has a unity 1) and every nonzero element in A has a multiplicative inverse. For general algebras this is not true.

The real *Cayley-Dickson* algebras $\mathbb{A}_n$, $n \geq 0$, are (nonassociative) real algebras with involution $*$, defined recursively as follows: $\mathbb{A}_0 = \mathbb{R}$ with trivial involution $a^* = a$, and $\mathbb{A}_n$ is the vector space $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ endowed with multiplication and involution defined by

$$(a,b)(c,d) = (ac - d^*b, da + bc^*),$$

$$(a,b)^* = (a^*, -b).$$

It is easy to see that $\mathbb{A}_n$ is unital (in fact, the unity of $\mathbb{A}_n$ is $(1,0)$ where 1 is the unity of $\mathbb{A}_{n-1}$), $x + x^*$ and $xx^* = x^*x$ are scalar multiplies of 1 for every $x \in \mathbb{A}_n$, and $\dim \mathbb{A}_n = 2^n$. Next, it is clear that $\mathbb{A}_1 = \mathbb{C}$, and one easily notices that $\mathbb{A}_2 = \mathbb{H}$, the *quaternions*. The next algebra in this process is $\mathbb{A}_3 = \mathbb{O}$, the *octonions*. For an excellent survey on octonions we refer the reader to [1]. Let us record here just a few basic properties of $\mathbb{O}$. First of all, $\mathbb{O}$ is an 8-dimensional division algebra. Denoting its basis by $\{1, e_1, \ldots, e_7\}$, the multiplication in $\mathbb{O}$ is determined by the following table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Note that the linear span of $1, e_1, e_2, e_3$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

It is well known that $\mathbb{O}$ is a division algebra which is not associative. However, it is "almost" associative - namely, it is alternative. Recall that an algebra A is said to be *alternative* if

$$x^2y = x(xy) \quad \text{and} \quad yx^2 = (yx)x \tag{1}$$

holds for all $x, y \in A$. Incidentally, Artin's theorem says that this is equivalent to the condition that any two elements generate an associative subalgebra [20, p. 36]. We shall need the identities from (1) in their linearized forms:

$$(xz + zx)y = x(zy) + z(xy), \; y(xz + zx) = (yx)z + (yz)x. \tag{2}$$

Let us also record the so-called middle Moufang identity which, as one easily checks (see, e.g., [20, p. 35]), holds in every alternative algebra:

$$(xy)(zx) = x(yz)x. \tag{3}$$

With regard to the right-hand side of (3) it should be pointed out that alternative algebras are flexible, i.e., $x(yx) = (xy)x$ holds (after all, this follows from Artin's theorem), and therefore there is a convention to write xyx instead of $(xy)x$ or $x(yx)$.

The next algebra obtained by the Cayley-Dickson process is the 16-dimensional algebra $\mathbb{A}_4 = \mathbb{S}$, the *sedenions*. Let $\{1, e_1, \ldots, e_{15}\}$ be a basis of $\mathbb{S}$. This is the multiplication table for $\mathbb{S}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

The sedenions have zero divisors and they are not an alternative algebra. Anyhow, we shall see that they are close enough to alternative division algebras, so that these approximate properties are "almost" characteristic for $\mathbb{S}$. Let us recall the definition of another notion needed for dealing with these properties.

An algebra A is said to be a *superalgebra* if it is $\mathbb{Z}_2$-graded, i.e., there exist linear subspaces A_i, $i \in \mathbb{Z}_2$, such that $A = A_0 \oplus A_1$ and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. We call A_0 an *even* and A_1 an *odd* part of A. Elements in $A_0 \cup A_1$ are said to be *homogeneous*. Note that if A is unital, then $1 \in A_0$.

Cayley-Dickson algebras possess a natural superalgebra structure. Indeed, $A = \mathbb{A}_n$ becomes a superalgebra by defining $A_0 = \mathbb{A}_{n-1} \times 0$ and $A_1 = 0 \times \mathbb{A}_{n-1}$. This simple observation is the concept behind the contents of Section 5.

The algebras $\mathbb{A}_n$, $n \geq 4$, are not alternative, but at least they have certain nonscalar elements that share many properties with elements in alternative algebras: these are scalar multiples of the element $e = (0, 1)$, where 1 is of course the unity of $\mathbb{A}_{n-1}$ (see e.g. [2, Section 5]). Let us point out only one property that is sufficient for our purposes: e satisfies $x^2e = x(xe)$ for all $x \in \mathbb{A}_n$. This can be easily verified. Moreover, this property is "almost" characteristic for e: only elements in the linear span of 1 and e satisfy this identity for every x [8, Lemma 1.2] (the authors are thankful to Alberto Elduque for drawing their attention to this result). Now, let us call an element a in an arbitrary nonassociative algebra A an *alter-scalar* if a is not a scalar and satisfies $x^2a = x(xa)$ holds for all $x \in A$. (A similar, but not exactly the same notion of a strongly alternative element was defined in [17]. There is also a standard notion of an alternative element defined through the condition $a^2x = a(ax)$ for every x, but this is too weak for our goals). What is important for us is that $\mathbb{S}$ contains alter-scalars. With respect to the notation introduced above, these are nonzero scalar multiplies of e_8. Thus, the standard basis of $\mathbb{S}$ has an element that is in some sense "better" than the others. This does not seem to be the case with the preceding Cayley-Dickson algebras.

Next we recall that an algebra A is said to be *quadratic* if it is unital and the elements $1, x, x^2$ are linearly dependent for every $x \in A$. Thus, for every $x \in A$ there exist $t(x), n(x) \in \mathbb{R}$ such that $x^2 - t(x)x + n(x) = 0$. Obviously, $t(x)$ and $n(x)$ are uniquely determined if $x \notin \mathbb{R}$. Setting $t(\lambda) = 2\lambda$ and $n(\lambda) = \lambda^2$ for $\lambda \in \mathbb{R}$, we can then consider t and n as maps from A into $\mathbb{R}$ (the reason for this definition is that in this way t becomes a linear functional, but we shall not need this). We call $t(x)$ and $n(x)$ the *trace* and the *norm* of x, respectively. For some elementary properties of quadratic algebras, a characterization of quadratic alternative algebras, and further references we refer to [9].

From $x^2 - (x+x^*)x + x^*x = 0$ we see that all algebras $\mathbb{A}_n$ are quadratic. Further, every real division algebra A that is algebraic and power-associative (this means that every subalgebra generated by one element is associative) is automatically quadratic. Indeed, if $x \in A$ then there exists a nonzero polynomial $f(X) \in \mathbb{R}[X]$ such that $f(x) = 0$. Writing $f(X)$ as the product of linear and quadratic polynomials in $\mathbb{R}[X]$ it follows that $p(x) = 0$ for some $p(X) \in \mathbb{R}[X]$ of degree 1 or 2. In particular, algebraic alternative (and hence associative) real division algebras are quadratic.

Finally, if A is a real unital algebra, i.e., an algebra over $\mathbb{R}$ with unity 1, then we shall follow a standard convention and identify $\mathbb{R}$ with $\mathbb{R}1$; thus we shall write λ for $\lambda 1$, where $\lambda \in \mathbb{R}$.

3. Frobenius' and Zorn's theorems

Our first lemma is well known. It describes one of the basic properties of quadratic algebras. We give the proof for the sake of completness.

Lemma 3.1. *Let A be a quadratic real algebra. Then $U = \{u \in A\backslash\mathbb{R} \mid u^2 \in \mathbb{R}\} \cup \{0\}$ is a linear subspace of A, $uv + vu \in \mathbb{R}$ for all $u, v \in U$, and $A = \mathbb{R} \oplus U$.*

Proof. Obviously, U is closed under scalar multiplication. We have to show that $u, v \in U$ implies $u + v \in U$. If $u, v, 1$ are linearly dependent, then one easily notices that already u and v are dependent, and the result follows. Thus, let $u, v, 1$ be independent. We have $(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2 \in \mathbb{R}$. On the other hand, as A is quadratic there exist $\lambda, \mu \in \mathbb{R}$ such that $(u + v)^2 - \lambda(u + v) \in \mathbb{R}$ and $(u-v)^2-\mu(u-v) \in \mathbb{R}$, and hence $\lambda(u+v)+\mu(u-v) \in \mathbb{R}$. However, the independence of $1, u, v$ implies $\lambda + \mu = \lambda - \mu = 0$, so that $\lambda = \mu = 0$. This proves that $u \pm v \in U$. Thus U is indeed a subspace of A. Accordingly, $uv+vu = (u+v)^2 - u^2 - v^2 \in \mathbb{R}$ for all $u, v \in U$. Finally, if $a \in A \setminus \mathbb{R}$, then $a^2 - \nu a \in \mathbb{R}$ for some $\nu \in \mathbb{R}$, and therefore $u = a - \frac{\nu}{2} \in U$; thus, $a = \frac{\nu}{2} + u \in \mathbb{R} \oplus U$. ☐

Remark 3.2. If A is additionally a division algebra, then every nonzero $u \in U$ can be written as $u = \alpha v$ with $\alpha \in \mathbb{R}$ and $v^2 = -1$. Indeed, since $u^2 \in \mathbb{R}$ and since u^2 cannot be ≥ 0 (otherwise $(u - \alpha)(u + \alpha) = u^2 - \alpha^2$ would be 0 for some $\alpha \in \mathbb{R}$) we have $u^2 = -\alpha^2$ with $0 \neq \alpha \in \mathbb{R}$. Thus, $v = \alpha^{-1}u$ is a desired element.

Note that by $\langle u, v\rangle = -\frac{1}{2}(uv + vu)$ one defines an inner product on U if A is a division algebra. The next lemma therefore deals with nothing but the Gram-Schmidt process. Nevertheless, we give the proof.

Lemma 3.3. *Let A be a quadratic real division algebra, and let U be as in Lemma 3.1. Suppose $e_1, \ldots, e_k \in U$ are such that $e_i^2 = -1$ for all $i \leq k$ and $e_ie_j = -e_je_i$ for all $i, j \leq k$, $i \neq j$. If U is not equal to the linear span of $e_1, \ldots, e_k$, then there exists $e_{k+1} \in U$ such that $e_{k+1}^2 = -1$ and $e_ie_{k+1} = -e_{k+1}e_i$ for all $i \leq k$.*

Proof. Pick $u \in U$ that is not contained in the linear span of $e_1, \dots, e_k$, and set $\alpha_i = \frac{1}{2}(ue_i + e_i u) \in \mathbb{R}$ (by Lemma 3.1). Note that $v = u + \alpha_1 e_1 + \ldots + \alpha_k e_k$ satisfies $e_i v = -ve_i$ for all $i \leq k$. Let e_{k+1} be a scalar multiple of v such that $e_{k+1}^2 = -1$ (Remark 3.2). Then e_{k+1} has all desired properties. □

Theorem 3.4. (Frobenius' theorem) *An algebraic associative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. As pointed out at the end of Section 2, A is quadratic. We may assume that $n = \dim A \geq 2$. By Remark 3.2 we can fix $i \in A$ such that $i^2 = -1$. Thus, $A \cong \mathbb{C}$ if $n = 2$. Let $n > 2$. By Lemma 3.3 there is $j \in A$ such that $j^2 = -1$ and $ij = -ji$. Set $k = ij$. Now one immediately checks that $k^2 = -1$, $ki = j = -ik$, $jk = i = -kj$, and i, j, k are linearly independent. Therefore A contains a subalgebra isomorphic to $\mathbb{H}$. It remains to show that n is not > 4. If it was, then by Lemma 3.3 there would exist $e \in A$ such that $e \neq 0$, $ei = -ie$, $ej = -je$, and $ek = -ke$. However, from the first two identities we infer $eij = -iej = ije$; since $ij = k$, this contradicts the third identity. □

In standard graduate algebra textbooks one can find different proofs of Frobenius' theorem. In some of them the advanced theory is used, but there are also such that use only elementary tools, e.g., [11] and [15]. The proof in [11] is actually based on similar ideas than our proof, but it is considerably lengthier. The one in [15] (which is based on [18]) is different, and also short.

We believe that our proof, consisting of four simple steps (Lemma 3.1, Remark 3.2, Lemma 3.3, and the final proof), should be easily understandable to undergraduate students. Some of these steps, especially both lemmas, are of independent interest.

We now switch to the proof of Zorn's theorem. We need a simple lemma:

Lemma 3.5. *Let A be an alternative algebra, and let $e_1, \dots, e_k \in A$ be such that $e_i e_j \in \{e_1, \dots, e_k\}$ whenever $i \neq j$. If $w \in A$ is such that $e_i w = -we_i$ for every i, then $(e_i e_j)w = -e_i(e_j w)$ and $w(e_i e_j) = -(we_i)e_j$ whenever $i \neq j$.*

Proof. Just set $x = e_i$, $y = e_j$, and $z = w$ in (2), and the result follows. □

Theorem 3.6. (Zorn's theorem) *An algebraic alternative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. Since a subalgebra generated by two elements is associative, the first part of the proof of Theorem 3.4 remains unchanged in the present context. We may therefore assume that A contains a copy of $\mathbb{H}$ and that $n = \dim A > 4$. Let us just change the notation and write $e_1 = i$, $e_2 = j$, and $e_3 = k$. By Lemma 3.3 there exists $e_4 \in A$ such that $e_4^2 = -1$ and $e_4 e_i = -e_i e_4$ for $i = 1, 2, 3$. Now define $e_5 = e_1 e_4$, $e_6 = e_2 e_4$, $e_7 = e_3 e_4$. Using the alternativity and anticommutativity relations we see that

$$e_5^2 = e_6^2 = e_7^2 = -1,$$

$$e_1 e_5 = -e_5 e_1 = e_2 e_6 = -e_6 e_2 = e_3 e_7 = -e_7 e_3 = -e_4,$$

$$e_4 e_5 = -e_5 e_4 = e_1,\ e_4 e_6 = -e_6 e_4 = e_2,\ e_4 e_7 = -e_7 e_4 = e_3.$$

Further, using (3) we obtain

$$e_5 e_6 = -e_6 e_5 = -e_3,\ e_6 e_7 = -e_7 e_6 = -e_1,\ e_7 e_5 = -e_5 e_7 = -e_2.$$

Finally, use Lemma 3.5 with $k = 3$ and $w = e_4$, and note that the resulting identites yield the rest of the multiplication table.

It is easy to see that $1, e_1, \dots, e_7$ are linearly independent. Indeed, by taking squares we first see that $\sum_{i=1}^{7} \lambda_i e_i$ cannot be a nonzero scalar; if $\sum_{i=1}^{7} \lambda_i e_i = 0$, then after multiplying this relation with e_i we get $\lambda_i = 0$. Thus, we have showed that A contains $\mathbb{O}$.

It remains to show that $n = 8$. Suppose $n > 8$. Then, by Lemma 3.3, there exists $f \in A$ such that $f \neq 0$ and $fe_i = -e_i f$, $1 \leq i \leq 7$. Lemma 3.5 tells us that f also satisfies $(e_i e_j)f = -e_i(e_j f)$ and $f(e_i e_j) = -(fe_i)e_j$ for $i \neq j$. Accordingly,

$$e_1(e_2(e_4 f)) = -e_1((e_2 e_4)f) = -e_1(e_6 f) = (e_1 e_6)f = -e_7 f. \tag{4}$$

Note that for $1 \leq i \leq 3$ we have

$$e_i(e_4 f) = -(e_i e_4)f = f(e_i e_4) = -f(e_4 e_i) = (fe_4)e_i = -(e_4 f)e_i.$$

This makes it possible for us to apply Lemma 3.5 for $k = 3$ and $w = e_4 f$. In particular this gives $(e_1 e_2)(e_4 f) = -e_1(e_2(e_4 f))$. Consequently,

$$e_1(e_2(e_4 f)) = -e_3(e_4 f) = (e_3 e_4)f = e_7 f,$$

contradicting (4). □

Remark 3.7. From the first part of the proof we see that if an alternative (not necessarily a division) real algebra A contains a copy of $\mathbb{H}$ and $\dim A > 4$, then it also contains a copy of $\mathbb{O}$.

Classical versions of Frobenius' and Zorn's theorems deal with finite dimensional algebras rather than with (slightly more general) algebraic ones. Our method, however, yields these more general versions for free. But actually we shall need the more general version of Zorn's theorem in Section 5.

We cannot claim that any of the arguments given in this section is entirely original. After finding these proofs we have realized, when searching the literature, that many of these ideas appear in different texts. But to the best of our knowledge nobody has compiled these arguments in the same way that leads to short and direct proofs of theorems by Frobenius and Zorn. Therefore we hope and believe that this section is of some value.

4. Locally complex algebras

As already mentioned, we define a *locally complex algebra* as a real unital algebra A such that every $a \in A \setminus \mathbb{R}$ generates a subalgebra isomorphic to $\mathbb{C}$. A locally complex algebra A is obviously quadratic. We can therefore consider the trace $t(a)$ and the norm $n(a)$ of each $a \in A$.

Lemma 4.1. *The following conditions are equivalent for a real unital algebra A:*

- (i) *A is locally complex;*
- (ii) *every $0 \neq a \in A$ has a multiplicative inverse lying in $\mathbb{R}a + \mathbb{R}$;*
- (iii) *A is quadratic and A has no nontrivial idempotents or square-zero elements;*
- (iv) *A is quadratic and $n(a) > 0$ for every $0 \neq a \in A$.*

Moreover, if $2 \leq \dim A = n < \infty$, then (i)-(iv) *are equivalent to*

- (v) *A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_i e_j = -e_j e_i$ for all $i \neq j$.*

Proof. It is easy to see that (i)$\Longrightarrow$ (ii) and (ii)$\Longrightarrow$ (iii). Suppose A is quadratic and $n(a) \leq 0$ for some $0 \neq a \in A$. Then $a \notin \mathbb{R}$. Therefore also $b = a - \frac{t(a)}{2} \notin \mathbb{R}$. Note that $b^2 \geq 0$. If $b^2 = 0$, then A has a nontrivial nilpotent. If $b^2 > 0$, i.e., $b^2 = \alpha^2$ for some $0 \neq \alpha \in \mathbb{R}$, then $e = \frac{1}{2}(1 - \alpha^{-1}b)$ is a nontrivial idempotent in A. Thus, (iii)$\Longrightarrow$ (iv). The proof of (iv)$\Longrightarrow$ (ii) is also straightforward. Therefore (ii)-(iv) are equivalent. Now assume (ii)-(iv) and pick $a \in A \setminus \mathbb{R}$. Then $b = a - \frac{t(a)}{2}$ satisfies $b^2 \in \mathbb{R}$. Just as in the argument above we see that b^2 cannot be ≥ 0. Hence $b^2 = -\alpha^2$ for some $\alpha \in \mathbb{R} \setminus \{0\}$, and so $i = \alpha^{-1}b$ satisfies $i^2 = -1$. This yields (i).

Finally, assume $2 \leq \dim A = n < \infty$. The implication (i)-(iv) $\Longrightarrow$ (v) follows from (the proof of) Lemma 3.3. Assuming (v) and writing $a \in A$ as $a = \lambda_0 + \sum_{i=1}^{n-1} \lambda_i e_i$, we see that $a^2 - t(a)a + n(a) = 0$ with $t(a) = 2\lambda_0$ and $n(a) = \sum_{i=0}^{n-1} \lambda_i^2$. Thus, (iv) holds. □

We can now list various examples of locally complex algebras.

Example 4.2. A quadratic real division algebra is locally complex.

Example 4.3. Let J_n be an n-dimensional real vector space, and let $\{1, e_1, \ldots, e_{n-1}\}$ be its basis. Define a multiplication in J_n so that 1 is of course the unity, and the others are multiplied according to $e_i e_j = -\delta_{ij}$. Then J_n is a locally complex algebra and simultaneously a Jordan algebra. Another way of representing J_n is by identifying it with $\mathbb{R} \times \mathbb{R}^{n-1}$, and defining multiplication by $(\lambda, u)(\mu, v) = (\lambda\mu - \langle u, v\rangle, \lambda v + \mu u)$, where $\langle\, . \,, . \,\rangle$ denotes the standard inner product on $\mathbb{R}^{n-1}$.

Example 4.4. A real unital algebra A is said to be *nicely normed* if there exists a linear map $* : A \to A$ such that $a^{**} = a$, $(ab)^* = b^* a^*$ for all $a, b \in A$, and $a + a^* \in \mathbb{R}$, $aa^* = a^* a > 0$ for all $0 \neq a \in A$ (cf. [1, p. 154]). These algebras form an important subclass of locally complex algebras. Namely, every element a in such an algebra A satisfies $a^2 - t(a)a + n(a) = 0$ with $t(a) = a + a^*$ and $n(a) = aa^*$, so that A is indeed locally complex. Note that $U = \{u \in A \setminus \mathbb{R} \mid u^2 \in \mathbb{R}\} \cup \{0\} = \{u \in A \mid u^* = -u\}$.

In particular, the Cayley-Dickson algebras $\mathbb{A}_n$ are nicely normed, and hence locally complex.

From Lemma 4.1 we can deduce the following characterization of finite dimensional nicely normed algebras.

Corollary 4.5. *let A be a real unital algebra. If $2 \leq \dim A = n < \infty$, then the following conditions are equivalent:*

(i) *A is nicely normed;*
(ii) *A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_i e_j = -e_j e_i \in \operatorname{span}\{e_1, \ldots, e_{n-1}\}$ for all $i \neq j$.*

Proof. Assume (i). By Lemma 4.1 (v) A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ that has all desired properties except that we do not know yet that $e_i e_j \in \operatorname{span}\{e_1, \ldots, e_{n-1}\}$. In view of the observation in Example 4.4 we have $\operatorname{span}\{e_1, \ldots, e_{n-1}\} = U = \{u \in A \mid u^* = -u\}$. Therefore, if $i \neq j$, $(e_i e_j)^* = e_j^* e_i^* = e_j e_i = -e_i e_j$, and hence $e_i e_j \in U$. Conversely, if (ii) holds, then we can define $*$ according to $1^* = 1$ and $e_i^* = -e_i$, and one easily checks that this makes A a nicely normed algebra. □

If A is a *commutative* finite dimensional locally complex algebra, then the e_i's from (v) in Lemma 4.1 must satisfy $e_i e_j = 0$ if $i \neq j$. This can be interpreted as follows.

Corollary 4.6. *Let A be a locally complex algebra with $2 \leq \dim A = n < \infty$. Then A is commutative if and only if $A \cong J_n$.*

Let A be an alternative real algebra. If A is an algebraic division algebra, then it is quadratic, and hence , as already mentioned, locally complex. Conversely, if A is locally complex, then by Lemma 4.1 (ii) for every $0 \neq a \in A$ there exist $\lambda, \mu \in \mathbb{R}$ such that $a(\lambda a + \mu) = 1$. Since A is alternative it follows that for every $y \in A$ the equation $ax = y$ has the solution $x = (\lambda a + \mu)y$. Similarly one solves the equation $xa = y$. Therefore A is an algebraic division algebra. Accordingly, Frobenius' and Zorn's theorem can be equivalently stated as follows.

Theorem 4.7. (Frobenius' and Zorn's theorems) *An associative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. An alternative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

As already mentioned in the introduction, this version of Frobenius' and Zorn's theorems indicates the direction in which these theorems can be generalized. We shall deal with this in the next section.

In the rest of this section we will classify locally complex algebras up to dimesion 4. Clearly, $\mathbb{R}$ and $\mathbb{C}$ are, up to an isomorphism, the only locally complex algebras of dimension ≤ 2.

We fix some notation. The members of $\mathbb{R} \times \mathbb{R}^2$ will be denoted by $(\lambda, x) = (\lambda, x_1, x_2)$ and the members of $\mathbb{R} \times \mathbb{R}^3$ by $(\lambda, x) = (\lambda, x_1, x_2, x_3)$. For each (ordered) pair $x, y \in \mathbb{R}^2$ we denote by $|x\ y|$ the 2×2 determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. The symbol $x \times y$ stands for the usual vector product (cross product) of $x, y \in \mathbb{R}^3$, while (x, y, z) denotes the scalar triple product $(x, y, z) = \langle x \times y, z\rangle$, $x, y, z \in \mathbb{R}^3$.

Let t, s be nonnegative real numbers. We denote by $A_{t,s}$ the 3-dimensional algebra $A_{t,s} = \mathbb{R} \times \mathbb{R}^2$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda y + \mu x + s|x\ y|e_1),$$

where $e_1 = (1, 0) \in \mathbb{R}^2$. It follows from Lemma 4.1 (v) that $A_{t,s}$ is a locally complex algebra. We will show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t, s) \in [0, \infty) \times [0, \infty)$ and that $A_{t,s}$ and $A_{t',s'}$ are not isomorphic whenever $(t, s) \neq (t', s')$. In short, we have the following classification theorem for 3-dimensional locally complex algebras.

Theorem 4.8. *The map* $(t, s) \mapsto A_{t,s}$, $t, s \geq 0$, *induces a bijection between* $[0, \infty) \times [0, \infty)$ *and isomorphism classes of 3-dimensional locally complex algebras.*

Proof. We first show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t, s) \in [0, \infty) \times [0, \infty)$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R} \times \mathbb{R}^2$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle, \lambda y + \mu x) + |x\ y|(t, z)$$

for some $(t, z) \in \mathbb{R} \times \mathbb{R}^2$. So, we may, and we will assume that A is this algebra. We have two possibilities; either $t \geq 0$, or $t < 0$. Let us consider only the second one; the case when $t \geq 0$ can be handled in a similar, but simpler way. Set $s = \|z\|$. There exists an orthogonal 2×2 matrix Q such that $Qz = -se_1$ and $\det Q = -1$. Observe that $|Qx\ Qy| = (\det Q)|x\ y| = -|x\ y|$ and $\langle Qx, Qy\rangle = \langle x, y\rangle$, $x, y \in \mathbb{R}^2$. We claim that the map $\varphi : A \to A_{|t|,s}$ given by $\varphi(\lambda, x) = (\lambda, Qx)$, $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^2$, is an isomorphism. Clearly, it is linear and bijective. Moreover, we have

$$\begin{aligned}\varphi((\lambda, x)\,(\mu, y)) &= \varphi((\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda y + \mu x + |x\ y|z)) \\ &= (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx - s|x\ y|e_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(\lambda, x)\,\varphi(\mu, y) &= (\lambda, Qx)\,(\mu, Qy) \\ = (\lambda\mu - \langle Qx, Qy\rangle &+ |t|\ |Qx\ Qy|, \lambda Qy + \mu Qx + s|Qx\ Qy|e_1) \\ &= (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx - s|x\ y|e_1).\end{aligned}$$

Hence, φ is an isomorphism. It remains to show that if $A_{t,s}$ and $A_{t',s'}$ are isomorphic for some $(t, s), (t', s') \in [0, \infty) \times [0, \infty)$, then $(t, s) = (t', s')$.

So, let $\varphi : A_{t,s} \to A_{t',s'}$ be an isomorphism. Then φ is linear and unital. In particular, $\varphi(\lambda, 0) = (\lambda, 0)$ for every $\lambda \in \mathbb{R}$. Furthermore, we have

$$\{(0, x) \in A_{t,s} \mid x \in \mathbb{R}^2\} = \{u \in A_{t,s} \mid u^2 \in \mathbb{R} \text{ and } u \notin \mathbb{R}\} \cup \{0\}.$$

It follows that

$$\varphi(\lambda, x) = (\lambda, Qx)$$

for some linear map $Q : \mathbb{R}^2 \to \mathbb{R}^2$. From

$$\begin{aligned}(\lambda^2 - \|Qx\|^2, 2\lambda Qx) &= (\lambda, Qx)^2 = (\varphi(\lambda, x))^2 \\ = \varphi((\lambda, x)^2) = \varphi(\lambda^2 - \|x\|^2, 2\lambda x) &= (\lambda^2 - \|x\|^2, 2\lambda Qx)\end{aligned}$$

we get that $\|Qx\|^2 = \|x\|^2$ for every $x \in \mathbb{R}^2$. Thus, Q is orthogonal. The equation

$$\varphi((\lambda, x)\,(\mu, y)) = \varphi(\lambda, x)\,\varphi(\mu, y)$$

can be rewritten as

$$\begin{aligned}&(\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx + s|x\ y|Qe_1)\\ &= (\lambda\mu - \langle x, y\rangle + t'(\det Q)\,|x\ y|, \lambda Qy + \mu Qx + s'(\det Q)\,|x\ y|e_1).\end{aligned}$$

We conclude that $t = t'\det Q$ and $sQe_1 = s'(\det Q)e_1$. Applying the fact that $|\det Q| = 1$ and $\|Qe_1\| = \|e_1\| = 1$ we get $|t| = |t'|$ and $|s| = |s'|$. As all t, t', s, s' are nonnegative, we have $t = t'$ and $s = s'$, as desired. $\square$

It follows directly from Corollary 4.5 that $A_{t,s}$ is nicely normed if and only if $t = 0$. So, the above statement shows that there is a natural bijection between $[0, \infty)$ and isomorphism classes of 3-dimensional nicely normed algebras.

The next result owes a lot to the paper [7] classifying 4-dimensional real quadratic division algebras. Our approach covers a more general class of real algebras. It is self-contained and completely elementary using just simple linear algebra tools.

We identify linear maps on $\mathbb{R}^3$ with 3×3 real matrices. Let M_3 denote the set of all 3×3 real matrices. For $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$ we write $(T, u) \sim (T', u')$ if and only if there exists an orthogonal 3×3 matrix Q such that $T' = (\det Q)QTQ^T$ and $u' = (\det Q)Qu$. It is clear that $\sim$ is an equivalence relation on $M_3 \times \mathbb{R}^3$. The set of equivalence classes will be denoted by $(M_3 \times \mathbb{R}^3)/\sim$.

For $T \in M_3$ and $u \in \mathbb{R}^3$ we denote by $A_{T,u}$ the 4-dimensional algebra $A_{T,u} = \mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y)).$$

As in the 3-dimensional case one can easily verify that $A_{T,u}$ is a locally complex algebra. We will show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$ and that $A_{T,u}$ and $A_{T',u'}$ are isomorphic if and only if $(T, u) \sim (T', u')$. In other words, we will prove the following.

Theorem 4.9. *The map $(T, u) \mapsto A_{T,u}$, $T \in M_3$, $u \in \mathbb{R}^3$, induces a bijection between $(M_3 \times \mathbb{R}^3)/\sim$ and isomorphism classes of 4-dimensional locally complex algebras.*

Proof. We will first show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle, \lambda y + \mu x) + S(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$$

for some linear map $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$. Observe that $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ can be decomposed into a direct sum of a linear functional on $\mathbb{R}^3$ and an endomorphism on $\mathbb{R}^3$. Recall that every linear functional on $\mathbb{R}^3$ can be represented in a unique way as an inner product with a fixed vector in $\mathbb{R}^3$. Finally, observe that the coordinates of the vector $(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$ are up to a permutation and a multiplication by ± 1 the coordinates of the vector product $x \times y$. Thus, A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y))$$

for some $u \in \mathbb{R}^3$ and some endomorphism T of $\mathbb{R}^3$. Hence, A is isomorphic to $A_{T,u}$, as desired.

Assume now that $A_{T,u}$ and $A_{T',u'}$ are isomorphic for some $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$. We have to show that $(T, u) \sim (T', u')$.

So, let $\varphi : A_{T,u} \to A_{T',u'}$ be an isomorphism. Exactly in the same way as in the 3-dimensional case we show that

$$\varphi(\lambda, x) = (\lambda, Qx)$$

for some orthogonal 3×3 matrix Q. The equation

$$\varphi((\lambda, x)\,(\mu, y)) = \varphi(\lambda, x)\,\varphi(\mu, y)$$

can be rewritten as

$$\begin{aligned}&(\lambda\mu-\langle x,y\rangle+(x,y,u),\lambda Qy+\mu Qx+QT(x\times y))\\=&(\lambda\mu-\langle x,y\rangle+(Qx,Qy,u'),\lambda Qy+\mu Qx+T'(Qx\times Qy)).\end{aligned}$$

We conclude that

$$(x,y,u)=(Qx,Qy,u')$$

and

$$QT(x\times y)=T'(Qx\times Qy)$$

for all $x,y\in\mathbb{R}^3$. As Q is orthogonal we have $Q(x\times y)=(\det Q)(Qx\times Qy)$, and consequently,

$$(x,y,u)=(\det Q)\,(x,y,Q^Tu')\quad\text{and}\quad QT(x\times y)=(\det Q)\,T'Q(x\times y),\quad x,y\in\mathbb{R}^3.$$

It follows that $u'=(\det Q)Qu$ and $T'=(\det Q)QTQ^T$, as desired.

Finally, if $(T,u)\sim(T',u')$ for some $T,T'\in M_3$ and $u,u'\in\mathbb{R}^3$ then there exists an orthogonal 3×3 matrix Q such that $T'=(\det Q)QTQ^T$ and $u'=(\det Q)Qu$. It is then straightforward to check that the map $\varphi:A_{T,u}\to A_{T',u'}$ defined by $\varphi(\lambda,x)=(\lambda,Qx)$, $(\lambda,x)\in A_{T,u}$, is an isomorphism. □

It is rather easy to verify that $A_{T,u}$ is nicely normed if and only if $u=0$. We will next show that $A_{T,u}$ is a division algebra if and only if $\langle Tx,x\rangle\neq 0$ for each nonzero $x\in\mathbb{R}^3$ (that is, the quadratic form $q(x)=\langle Tx,x\rangle$ is either positive definite, or negative definite). Indeed, assume first that $A_{T,u}$ is not a division algebra. Then

$$(\lambda\mu-\langle x,y\rangle+(x,y,u),\lambda y+\mu x+T(x\times y))=0$$

for some nonzero $(\lambda,x),(\mu,y)\in A_{T,u}$. In particular,

$$T(x\times y)=-\lambda y-\mu x.$$

Set $z=x\times y$. We have $z\neq 0$, since otherwise x and y are linearly dependent and therefore

- either $\lambda=0$ and then $\langle x,y\rangle=0$ and $\mu x=0$ which further yields that $(\lambda,x)=0$ or $(\mu,y)=0$, a contradiction; or
- $\mu=0$ which yields a contradiction in exactly the same way; or
- $\lambda\neq 0$ and $\mu\neq 0$ and then $y=-\mu\lambda^{-1}x$ and $\lambda\mu=\langle x,y\rangle$ yield $0<\lambda^2=-\langle x,x\rangle\leq 0$, a contradiction.

Hence, $z\neq 0$ and because z is orthogonal to both x and y we have $\langle Tz,z\rangle=0$.

To prove the other direction we assume that there exists $z\in\mathbb{R}^3$ with $\|z\|=1$ and $\langle Tz,z\rangle=0$. Then $Tz=-tw$ for some real number t and some $w\in\mathbb{R}^3$ with $w\perp z$ and $\|w\|=1$. There is a unique $v\in\mathbb{R}^3$ such that $z=w\times v$ and $v\perp w$. Set $s=-(w,v,u)$. Then $(0,w)$ and $(t,v-sw)$ are nonzero elements of $A_{T,u}$ whose product is equal to zero. Hence, $A_{T,u}$ is not a division algebra, as desired.

Following Dieterich's idea [7] we will now disscuss a geometric interpretation of the classification of 4-dimensional locally complex algebras. Let us start with a simple observation concerning 3×3 skew-symmetric matrices. If $x,y\in\mathbb{R}^3$ are any two vectors such that $x\times y=(c_1,c_2,c_3)$, then

$$R=\begin{bmatrix}0&c_3&-c_2\\-c_3&0&c_1\\c_2&-c_1&0\end{bmatrix}=xy^T-yx^T,$$

where x and y are represented as 3×1 matrices. If Q is any orthogonal matrix, then $QRQ^T=(Qx)(Qy)^T-(Qy)(Qx)^T$. As $Qx\times Qy=(\det Q)\,Q(x\times y)$, we have

$$Q\begin{bmatrix}0&c_3&-c_2\\-c_3&0&c_1\\c_2&-c_1&0\end{bmatrix}Q^T=\begin{bmatrix}0&d_3&-d_2\\-d_3&0&d_1\\d_2&-d_1&0\end{bmatrix},$$

where

$$\left[\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right] = (\det Q)\, Q \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right].$$

If we choose $Q \in SO(3)$ such that

$$\left[\begin{array}{c} 0 \\ 0 \\ \sqrt{c_1^2 + c_2^2 + c_3^2} \end{array}\right] = Q \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right],$$

then

$$QRQ^T = \left[\begin{array}{ccc} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where $d = \sqrt{c_1^2 + c_2^2 + c_3^2}$. In particular, $d = \|R\|$.

Any 3×3 matrix T can be uniquely decomposed into its symmetric and skew-symmetric part, $T = P + R$, $P = (1/2)(T + T^T)$, $R = (1/2)(T - T^T)$. If $T' = (\det Q)QTQ^T$ and $T' = P' + R'$ with P' symmetric and R' skew-symmetric, then $P' = (\det Q)QPQ^T$ and $R' = (\det Q)QRQ^T$. We will say that $A_{T,u}$ is of rank 3,2,1,0, respectively, if the symmetric part P of T is of rank 3,2,1,0, respectively. By the previous remark, two isomorphic algebras $A_{T,u}$ have the same rank.

Let us start with algebras $A_{T,u}$ of rank 3. We have two possibilities: either all eigenvalues of $P = T + T^T$ have the same sign, or P has both positive and negative eigenvalues. In the first case we will say that $A_{T,u}$ is an ellipsoid locally complex algebra of dimension 4, while in the second case we call $A_{T,u}$ a hyperboloid locally complex algebra of dimension 4. As we are interested in isomorphism classes we can use the fact that $A_{T,u}$ is isomorphic to $A_{-T,u}$ to restrict our attention to the case when all the eigenvalues of P are positive (the ellipsoid case) or to the case when two eigenvalues of P are positive and one is negative (the hyperboloid case). Once we have done this restriction two algebras $A_{T,u}$ and $A_{T',u'}$ of the above types are isomorphic if and only if $T' = QTQ^T$ and $u' = Qu$ for some $Q \in SO(3)$.

To consider isomorphism classes of hyperboloid locally complex algebras of dimension 4 (a 4-dimensional locally complex algebra is hyperboloid if it is isomorphic to some hyperboloid algebra $A_{T,u}$) we set $\tau = \{\delta \in \mathbb{R}^3 \,|\, \delta_1 \geq \delta_2 > 0 > \delta_3\}$ and $\kappa = \tau \times \mathbb{R}^3 \times \mathbb{R}^3$. The elements of κ will be called configurations. Each configuration consists of a hyperboloid $H_\delta = \{x \in \mathbb{R}^3 \,|\, \langle \Delta_\delta x, x\rangle = 1\}$ (a hyperboloid in principal axis form) and a pair of points. Here, Δ_δ is the diagonal matrix with the diagonal entries: $\delta_1, \delta_2, \delta_3$. The symmetry group of the hyperboloid H_δ is defined to be $G_\delta = \{Q \in SO(3) \,|\, Q\Delta_\delta Q^T = \Delta_\delta\}$ (the requirement that $\det Q = 1$ tells that we allow only symmetries that preserve the orientation). Note that this symmetry group consists of 4 elements whenever $\delta_1 > \delta_2$. Namely, in this case the symmetry group consists of the identity and all diagonal matrices with two eigenvalues -1 and one eigenvalue 1. The symmetry group is infinite if and only if the hyperboloid H_δ is circular, that is, $\delta_1 = \delta_2$. Two configurations (δ, u, c) and (δ', u', c') are said to be equivalent, $(\delta, u, c) \equiv (\delta', u', c')$, if and only if their hyperboloids coincide and their pairs of points lie in the same orbit under the operation of the symmetry group of the hyperboloid, that is, if and only if $\delta = \delta'$ and $(u', c') = (Qu, Qc)$ for some $Q \in G_\delta$. We denote by $\kappa/\equiv$ the set of equivalence classes of κ. We have a natural bijection between $\kappa/\equiv$ and the set of equivalence classes of hyperboloid locally complex algebras of dimension 4. Indeed, the bijection is induced by the map

$$(\delta, u, c) \mapsto A_{\Delta_\delta + R_c, u}$$

where

$$\Delta_\delta + R_c = \begin{bmatrix} \delta_1 & c_3 & -c_2 \\ -c_3 & \delta_2 & c_1 \\ c_2 & -c_1 & \delta_3 \end{bmatrix}.$$

Clearly, $A_{\Delta_\delta+R_c,u}$ is a hyperboloid locally complex algebra. We have to show that each hyperboloid algebra $A_{T,v}$ is isomorphic to some $A_{\Delta_\delta+R_c,u}$ and that $A_{\Delta_\delta+R_c,u}$ and $A_{\Delta_{\delta'}+R_{c'},u'}$ are isomorphic if and only if $(\delta, u, c) \equiv (\delta', u', c')$. The second statement is trivial. To verify the first one we write $T = P + R$ with P symmetric with two positive eigenvalues and R skew-symmetric. Then there exists $Q \in SO(3)$ such that $QPQ^T = \Delta_\delta$ for some $\delta \in \tau$. We have $QRQ^T = R_c$ for some $c \in \mathbb{R}^3$. Set $u = Qv$ to complete the proof.

In a similar fashion we can consider isomorphism classes of ellipsoid locally complex algebras of dimension 4. Note that a locally complex algebra $A_{T,u}$ is a division algebra if and only if it is an ellipsoid algebra. As above we can consider configurations which consist of an ellipsoid in principal axis form and a pair of points. To each such configuration there corresponds a 4-dimensional real division algebra and this correspondence induces a bijection between the equivalence classes of configurations (the equivalence being defined via the symmetry group of the ellipsoid) and the isomorphism classes of 4-dimensional real quadratic division algebras. We omit the details that can be found in [7]. It is clear that locally complex algebras of rank 2 are either elliptic cylinder algebras or hyperbolic cylinder algebras. We leave the details to the reader. In the same way one can classify also isomorphism classes of locally complex algebras of rank 1. Let us conclude with the detailed disscussion on 4-dimensional locally complex algebras of rank 0. By e_3 we denote $e_3 = (0, 0, 1) \in \mathbb{R}^3$. We define an equivalence relation on the set $[0, \infty) \times \mathbb{R}^3$ as follows: $(d, u), (d', u') \in [0, \infty) \times \mathbb{R}^3$ are said to be equivalent, $(d, u) \equiv (d', u')$, if either

- $d = d' = 0$ and $\|u\| = \|u'\|$; or
- $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3\rangle = \langle u', e_3\rangle$.

Note that the equivalence class of $(d, u) \in [0, \infty) \times \mathbb{R}^3$ with $d > 0$ contains infinitely many elements if u and e_3 are linearly independent, and is a singleton when u is a scalar multiple of e_3. There is a natural bijection between the isomorphism classes of 4-dimensional locally complex algebras of rank 0 and the set $([0, \infty) \times \mathbb{R}^3)/\equiv$. The bijection is induced by the map from $[0, \infty) \times \mathbb{R}^3$ which maps the pair (d, u), $d \geq 0$, $u \in \mathbb{R}^3$, into $A_{T_d,u}$ with

$$T_d = \begin{bmatrix} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $A_{T_d,u}$ is a locally complex algebra of rank 0 and one can easily verify that each 4-dimensional locally complex algebra of rank 0 is isomorphic to some $A_{T_d,u}$. It remains to show that $A_{T_d,u}$ and $A_{T_{d'},u'}$ are isomorphic if and only if $(d, u) \equiv (d', u')$. So, assume that $A_{T_d,u}$ and $A_{T_{d'},u'}$ are isomorphic for some $(d, u), (d', u') \in [0, \infty) \times \mathbb{R}^3$. Then there exists an orthogonal matrix Q such that $T_{d'} = (\det Q)QT_dQ^T$ and $u' = (\det Q)Qu$. In particular, $d' = \|T_{d'}\| = \|T_d\| = d$ and $\|u'\| = \|u\|$. If $d = 0$, then $d' = 0$, and hence, $(d, u) \equiv (d', u')$ in this special case. Therefore we may assume that $d = d' > 0$. From $T_{d'} = (\det Q)QT_dQ^T$ we conclude that $Qe_3 = (\det Q)e_3$. Consequently,

$$\langle u', e_3\rangle = \langle (\det Q)Qu, (\det Q)Qe_3\rangle = \langle u, e_3\rangle.$$

To prove the converse we assume that $(d, u) \equiv (d', u')$. We have one of the two possibilities and we will consider just the second one. So, assume that $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3\rangle = \langle u', e_3\rangle$. Then there exists an orthogonal matrix Q such

that $Qe_3 = e_3$ and $Qu = u'$. The orthogonal complement of e_3 and u is one-dimensional (if e_3 and u are linearly independent) or two-dimensional (if e_3 and u are linearly dependent). We have a freedom to choose the action of Q on the orthogonal complement of e_3 and u (of course, up to the requirement that Q is an orthogonal matrix). In particular, we can choose Q in such a way that $\det Q = 1$. It follows that $T_{d'} = QT_dQ^T$ and $u' = Qu$, as desired.

5. Super-alternative locally complex algebras

Let us call an algebra A a *super-alternative algebra* if it is $\mathbb{Z}_2$-graded, $A = A_0 \oplus A_1$, and the alternativity conditions (1) hold for all its homogeneous elements. Equivalently,

$$u^2x = u(ux),\ xu^2 = (xu)u \quad \text{for all } u \in A_i,\ i \in \mathbb{Z}_2,\ x \in A, \tag{5}$$

or, in the linearized form,

$$\begin{aligned}&(uv + vu)x = u(vx) + v(ux),\\ &x(uv + vu) = (xu)v + (xv)u \quad \text{for all } u, v \in A_i,\ i \in \mathbb{Z}_2\ ,\ x \in A.\end{aligned} \tag{6}$$

The notion of a super-alternative algebra should not be confused with the notion of an *alternative superalgebra*. The latter is defined through the alternativity of the Grassmann envelope of A. It turns out that nontrivial examples of alternative superalgebras exist only very exceptionally: prime alternative superalgebras of characteristic different from 2 and 3 are either associative or their odd part is zero [19]. As we shall see, super-alternative algebras are more easy to find.

Throughout this section A *will be a super-alternative locally complex algebra.* Our goal is to to classify all such algebras A. Obvious examples are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, as we can always take the trivial $\mathbb{Z}_2$-grading (the odd part is 0). Further, one can check by a straigtforward calculation that if $\mathbb{A}_{n-1}$ is an alternative algebra, then every $u \in (\mathbb{A}_{n-1} \times 0) \cup (0 \times \mathbb{A}_{n-1})$ satisfies (5) for every $x \in \mathbb{A}_n$. Therefore, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$ are super-alternative algebras with respect to the natural $\mathbb{Z}_2$-grading mentioned in Section 2. Of course, the important information for us in this context is that $\mathbb{S}$ is also a super-alternative locally complex algebra. As we shall see, besides $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and $\mathbb{S}$ only two more algebras must be added to the complete list of such algebras.

We continue by recording several simple but useful observations. First, the following special case of (6) will be often used:

(a) If $u, v \in A_i$, $i \in \mathbb{Z}_2$, are such that $uv + vu = 0$, then $u(vx) = -v(ux)$ and $(xu)v = -(xv)u$ for all $x \in A$.

If $v \in A_1$, then $v^2 \in A_0$; on the other hand, $v^2 = \lambda v + \mu$ for some $\lambda, \mu \in \mathbb{R}$. Since $v \notin A_0$, we must have $\lambda = 0$ and hence $v^2 = \mu \in \mathbb{R}$. Since A is locally complex, it follows that $\mu < 0$ if $v \neq 0$. Thus, we have

(b) If $0 \neq v \in A_1$, then there is $\alpha \in \mathbb{R}$ such that $(\alpha v)^2 = -1$.

Let $u \in A_0$ and $v \in A_1$ be such that $u^2 = v^2 = -1$. Using Lemma 3.1 we have $uv + vu \in \mathbb{R} \cap A_1 = 0$. Therefore $v(uv) = -v(vu) = -v^2u = u$. Next, $(uv)v = uv^2 = -u$. Similarly we see that $(uv)u = -u(uv) = v$. Finally, using (a) we get $(uv)(uv) = -(uv)(vu) = v((uv)u) = v^2 = -1$. We have proved:

(c) If $u \in A_0$ and $v \in A_1$ are such that $u^2 = v^2 = -1$, then $uv = -vu$, $v(uv) = -(uv)v = u$, $(uv)u = -u(uv) = v$, and $(uv)^2 = -1$.

Let u be a homogeneous element and suppose that $ux = 0$ for some $x \in A$. If $u \neq 0$, then by multiplying this identity from the left by $u - t(u)$ it follows from (5) that $n(u)x = 0$, and hence $x = 0$. Similarly, $xu = 0$ implies $x = 0$ if $u \neq 0$. Thus:

(d) Homogeneous elements are not zero divisors.

It is clear that our conditions on A imply that A_0 is a locally complex alternative algebra. Theorem 4.7 therefore tells us that A_0 is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. If $A_1 = 0$, then we get the desired conclusion that $A = A_0$ is one of the algebras from the expected list. Without loss of generality we may therefore assume that $A_1 \neq 0$. Given $0 \neq u \in A_1$, it follows from (d) that $x \mapsto ux$ is an injective linear map from A_0 into A_1; the same rule defines an injective linear map from A_1 into A_0. We may therefore conclude that

(e) $\dim A_0 = \dim A_1$.

In particular we now know that a super-alternative locally complex algebra must be finite dimensional. Moreover, its dimension can be only 1, 2, 4, 8, or 16.

We shall now consider separately each of the four possibilities concerning A_0.

Lemma 5.1. *If* $A_0 \cong \mathbb{R}$, *then* $A \cong \mathbb{C}$.

Proof. By (b) there is $i \in A_1$ with $i^2 = -1$, and hence $A \cong \mathbb{C}$ by (e). □

Lemma 5.2. *If* $A_0 \cong \mathbb{C}$, *then* $A \cong \mathbb{H}$.

Proof. We have $A_0 = \mathbb{R} \oplus \mathbb{R}i$ with $i^2 = -1$. By (b) we may pick $j \in A_1$ such that $j^2 = -1$. Setting $k = ij \in A_1$ it follows from (c) that A contains a copy of $\mathbb{H}$. However, in view of (e) we actually have $A \cong \mathbb{H}$. □

Let us now introduce another (an unexpected one for us) example of a super-alternative locally complex algebra. Let $\widetilde{\mathbb{O}}$ be the 8-dimensional algebra with basis $\{1, f_1, \dots, f_7\}$ and multiplication table

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	f_7	$-f_6$
f_2	$-f_3$	-1	f_1	f_6	$-f_7$	$-f_4$	f_5
f_3	f_2	$-f_1$	-1	f_7	f_6	$-f_5$	$-f_4$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3
f_5	f_4	f_7	$-f_6$	$-f_1$	-1	f_3	$-f_2$
f_6	$-f_7$	f_4	f_5	$-f_2$	$-f_3$	-1	f_1
f_7	f_6	$-f_5$	f_4	$-f_3$	f_2	$-f_1$	-1

Lemma 5.3. $\widetilde{\mathbb{O}}$ *is a super-alternative locally complex algebra with zero divisors and without alter-scalar elements (and hence* $\widetilde{\mathbb{O}} \ncong \mathbb{O}$*).*

Proof. The fact that $\widetilde{\mathbb{O}}$ is locally complex follows from Lemma 4.1 (v). Let $\widetilde{\mathbb{O}}_0$ be the linear span of $1, f_1, f_2, f_3$, and let $\widetilde{\mathbb{O}}_1$ be the linear span of f_4, f_5, f_6, f_7. Then $\widetilde{\mathbb{O}}$ becomes a superalgebra with the even part $\widetilde{\mathbb{O}}_0 \cong \mathbb{H}$. From the way we shall arrive at $\widetilde{\mathbb{O}}$ in the next proof it is not really surprising that $\widetilde{\mathbb{O}}$ is super-alternative. But we used Mathematica for the actual checking that this is indeed true. Note that $(f_1 - f_4)(f_3 - f_6) = 0$, so that $\widetilde{\mathbb{O}}$ has zero divisors. Let $a \in \widetilde{\mathbb{O}}$ be such that $x^2 a = x(xa)$ for all $x \in \widetilde{\mathbb{O}}$. From $(f_i + f_j)^2 a = (f_i + f_j)((f_i + f_j)a)$, together with $f_i(f_i a) = f_j(f_j a) = -a$, it follows that $f_i(f_j a) + f_j(f_i a) = 0$ whenever $i \neq j$. Writing $a = \lambda_0 + \sum_{k=1}^{7} \lambda_k f_k$ we thus have

$$\sum_{k=1}^{7} \lambda_k \Big(f_i(f_j f_k) + f_j(f_i f_k) \Big) = 0 \quad \text{whenever } i \neq j. \tag{7}$$

Chosing $i = 1$ and $j = 4$ it follows that $\lambda_2 = \lambda_3 = \lambda_6 = \lambda_7 = 0$. Chosing, for example, $i = 2$ and $j = 7$ we further get $\lambda_1 = \lambda_4 = 0$, and chosing $i = 3$ and $j = 4$ finally leads to $\lambda_5 = 0$. Therefore $a = \lambda_0$ is a scalar. □

Lemma 5.4. *If* $A_0 \cong \mathbb{H}$, *then* $A \cong \mathbb{O}$ *or* $A \cong \widetilde{\mathbb{O}}$.

Proof. Let $\{1, i, j, k\}$ be a basis of A_0 where these elements have the usual meaning. Pick $f \in A_1$ with $f^2 = -1$. Then f anticommutes with i, j, k by (c). It is clear that $\{f, if, jf, kf\}$ is a basis of A_1. We claim that all elements in this basis pairwise anticommute. It is easy to see that f anticommutes with each of if, jf, kf. Using (a) repeatedly we obtain $(if)(jf) = -(i(jf))f = (j(if))f = -(jf)(if)$. Other identities can be checked analogously.

Since $i(jf) \in A_1$, we have

$$i(jf) = \lambda_1 f + \lambda_2 if + \lambda_3 jf + \lambda_4 kf \tag{8}$$

for some $\lambda_i \in \mathbb{R}$. From (a) we infer that $(i(jf))f = -(if)(jf)$. Similarly, using (a) and (c) we get

$$f(i(jf)) = -f((jf)i) = (jf)(fi) = -(jf)(if) = (if)(jf).$$

The last two identities show that $i(jf)$ anticommutes with f. Consequently, anticommuting (8) with f it follows that $\lambda_1 = 0$. A similar arguing shows that $i(jf)$ anticommutes with both if and jf, which leads to $\lambda_2 = \lambda_3 = 0$. Note that (c) implies that the squares of both kf and $i(jf)$ are equal -1. But then $\lambda_4^2 = 1$, i.e., $\lambda_4 = 1$ or $\lambda_4 = -1$. If $\lambda_4 = 1$, i.e., $i(jf) = kf$, then we set $f_1 = i$, $f_2 = j$, $f_3 = k$, $f_4 = f$, $f_5 = if$, $f_6 = jf$, and $f_7 = kf$. Using the information we have, it is now just a matter of a routine calculation to verify that $A \cong \widetilde{\mathbb{O}}$. Since we know that $\mathbb{O}$ is a super-alternative locally complex algebra, the other possibility $\lambda_4 = -1$ can lead only to $A \cong \mathbb{O}$. $\square$

The 16-dimensional analogue of $\widetilde{\mathbb{O}}$ is the algebra which we denote by $\widetilde{\mathbb{S}}$ and define as follows: if $\{1, f_1, \ldots, f_{15}\}$ is its basis, then the multiplication table is

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	$-f_7$	f_6	f_9	$-f_8$	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}
f_2	$-f_3$	-1	f_1	f_6	f_7	$-f_4$	$-f_5$	f_{10}	f_{11}	$-f_8$	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$
f_3	f_2	$-f_1$	-1	f_7	$-f_6$	f_5	$-f_4$	f_{11}	$-f_{10}$	f_9	$-f_8$	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3	f_{12}	f_{13}	f_{14}	$-f_{15}$	$-f_8$	$-f_9$	$-f_{10}$	f_{11}
f_5	f_4	$-f_7$	f_6	$-f_1$	-1	$-f_3$	f_2	f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	$-f_8$	f_{11}	f_{10}
f_6	f_7	f_4	$-f_5$	$-f_2$	f_3	-1	$-f_1$	f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	$-f_8$	$-f_9$
f_7	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1	f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	$-f_8$
f_8	$-f_9$	$-f_{10}$	$-f_{11}$	$-f_{12}$	$-f_{13}$	$-f_{14}$	$-f_{15}$	-1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_9	f_8	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}	$-f_1$	-1	$-f_3$	f_2	$-f_5$	f_4	$-f_7$	f_6
f_{10}	f_{11}	f_8	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$	$-f_2$	f_3	-1	$-f_1$	$-f_6$	f_7	f_4	$-f_5$
f_{11}	$-f_{10}$	f_9	f_8	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$	$-f_3$	$-f_2$	f_1	-1	f_7	f_6	$-f_5$	$-f_4$
f_{12}	f_{13}	f_{14}	$-f_{15}$	f_8	$-f_9$	$-f_{10}$	f_{11}	$-f_4$	f_5	f_6	$-f_7$	-1	$-f_1$	$-f_2$	f_3
f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	f_8	f_{11}	f_{10}	$-f_5$	$-f_4$	$-f_7$	$-f_6$	f_1	-1	f_3	f_2
f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	f_8	$-f_9$	$-f_6$	f_7	$-f_4$	f_5	f_2	$-f_3$	-1	$-f_1$
f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	f_8	$-f_7$	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1

The proof of the next lemma is similar to that of Lemma 5.3. Therefore we omit details.

Lemma 5.5. *$\widetilde{\mathbb{S}}$ is a super-alternative locally complex algebra without alter-scalar elements (and hence $\widetilde{\mathbb{S}} \ncong \mathbb{S}$).*

The final lemma is similar to Lemma 5.4, but the proof is somewhat more complicated. One of the problems that we have to face in this proof is that we do not have a complete freedom in the selection of an element playing the role of f from the proof of Lemma 5.4. While f was an arbitrary element in A_1 with square -1, now we shall have to find a special one.

Lemma 5.6. *If $A_0 \cong \mathbb{O}$, then $A \cong \mathbb{S}$ or $A \cong \widetilde{\mathbb{S}}$.*

Proof. Let $\{1, e_1, \ldots, e_7\}$ be a basis of A_0 whose multiplication table is given in Section 2. We begin with three claims needed for future reference.

CLAIM 1: Let $i, j \in \{1, 2, \ldots, 7\}$, $i \neq j$. If $p \in A_1$, then $q = p + (e_i e_j)(e_i(e_j p))$ satisfies $(e_i e_j)q = -e_i(e_j q)$.

Indeed, by (5) we have $(e_ie_j)q = (e_ie_j)p - e_i(e_jp)$, while using (a) and (5) we get

$$\begin{aligned} e_i(e_jq) &= e_i(e_jp) + e_i(e_j((e_ie_j)(e_i(e_jp)))) = e_i(e_jp) - e_i((e_ie_j)(e_j(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_i(e_j(e_i(e_jp)))) = e_i(e_jp) - (e_ie_j)(e_j(e_i(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_j(e_jp)) = e_i(e_jp) - (e_ie_j)p, \end{aligned}$$

so that $(e_ie_j)q = -e_i(e_jq)$.

Claim 2: Let $i, j, k \in \{1, 2, \dots, 7\}$ be such that e_i, e_j, e_ie_j, e_k are linearly independent, and let $s \in A_1$ be such that $(e_ie_j)s = -e_i(e_js)$. Then $t = s + (e_ie_k)(e_i(e_ks))$ also satisfies $(e_ie_j)t = -e_i(e_jt)$.

(Let us add that (a) implies $t = s + (e_ke_i)(e_k(e_is))$, and that $(e_ie_j)z = -e_i(e_jz)$ is equivalent to $(e_je_i)z = -e_j(e_iz)$; the order of indices is thus irrelevant.)

Indeed, by now already familiar arguing we have

$$\begin{aligned} (e_ie_j)t &= (e_ie_j)s + (e_ie_j)((e_ie_k)(e_i(e_ks))) = (e_ie_j)s - (e_ie_k)((e_ie_j)(e_i(e_ks))) \\ &= (e_ie_j)s + (e_ie_k)(e_i((e_ie_j)(e_ks))) = (e_ie_j)s - (e_ie_k)(e_i(e_k((e_ie_j)s))) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_i(e_k(e_i(e_js))))\big) = -\big(e_i(e_js) + (e_ie_k)(e_k(e_i(e_i(e_js))))\big) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_k(e_js))\big) = -\big(e_i(e_js) + e_i(e_i((e_ie_k)(e_k(e_js))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_i(e_k(e_js))))\big) = -\big(e_i(e_js) + e_i((e_ie_k)(e_i(e_j(e_ks))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_j(e_i(e_ks))))\big) = -\big(e_i(e_js) + e_i(e_j((e_ie_k)(e_i(e_ks))))\big) \\ &= -e_i(e_jt). \end{aligned}$$

Claim 3: Let $i, j, k \in \{1, 2, \dots, 7\}$, $i \neq j$, and let $\epsilon \in \mathbb{R}$ and $w \in A_1$ be such that $(e_ie_j)w = \epsilon e_i(e_jw)$. Set $u = e_kw$. If $k \in \{i, j\}$, then $(e_ie_j)u = \epsilon e_i(e_ju)$, and if $k \notin \{i, j\}$, then $(e_ie_j)u = -\epsilon e_i(e_ju)$.

If $k \in \{i, j\}$, then we may assume $k = j$ without loss of generality. We have

$$(e_ie_j)(u) = (e_ie_j)(e_jw) = -e_j((e_ie_j)w) = -\epsilon e_j(e_i(e_jw)) = \epsilon e_i(e_ju).$$

If $k \notin \{i, j\}$, then we have

$$\begin{aligned} (e_ie_j)(u) &= (e_ie_j)(e_kw) = -e_k((e_ie_j)w) \\ &= -\epsilon e_k(e_i(e_jw)) = \epsilon e_i(e_k(e_jw)) = -\epsilon e_i(e_ju). \end{aligned}$$

After establishing these auxiliary claims, we now begin the actual proof by picking a nonzero $u \in A_1$. As mentioned above, an arbitrary chosen u may not be the right choice, so we have to "remedy" it. Let $v' = u + (e_1e_2)(e_1(e_2u)) \in A_1$. By Claim 1, v' satisfies $(e_1e_2)v' = -e_1(e_2v')$. If $v' = 0$, then we have $(e_1e_2)u = e_1(e_2u)$. But then $v'' = e_3u$ satisfies $(e_1e_2)v'' = -e_1(e_2v'')$ by Claim 3. Thus, in any case there is a nonzero $v \in A_1$ such that

$$(e_1e_2)v = -e_1(e_2v).$$

Now consider $w' = v + (e_1e_4)(e_1(e_4v))$. By Claim 1 we have $(e_1e_4)w' = -e_1(e_4w')$, and by Claim 2 we have $(e_1e_2)w' = -e_1(e_2w')$. If $w' = 0$, then $(e_1e_4)v = e_1(e_4v)$. But then $w'' = e_2v$ satisfies $(e_1e_2)w'' = -e_1(e_2w'')$ and $(e_1e_4)w'' = -e_1(e_4w'')$. Thus, there exists a nonzero $w \in A_1$ satisfying

$$(e_1e_2)w = -e_1(e_2w), \ (e_1e_4)w = -e_1(e_4w).$$

We now repeat the same procedure with respect to e_2 and e_4. That is, we introduce $x' = w + (e_2e_4)(e_2(e_4w))$, and apply Claims 1 and 2 to conclude that $(e_1e_2)x' = -e_1(e_2x')$, $(e_1e_4)x' = -e_1(e_4x')$, and $(e_2e_4)x' = -e_2(e_4x')$. If $x' = 0$, then $(e_2e_4)w = e_2(e_4w)$, and therefore Claim 3 tells us that $(e_1e_2)x'' = -e_1(e_2x'')$, $(e_1e_4)x'' = -e_1(e_4x'')$, and $(e_2e_4)x'' = -e_2(e_4x'')$, where $x'' = e_1w$. In any case we have found a a nonzero $x \in A_1$ satisfying

$$(e_1e_2)x = -e_1(e_2x), \ (e_1e_4)x = -e_1(e_4x), \ (e_2e_4)x = -e_2(e_4x).$$

Considering $y' = x + (e_3e_4)(e_3(e_4x))$ we see from Claim 2 that $(e_1e_4)y' = -e_1(e_4y')$ and $(e_2e_4)y' = -e_2(e_4y')$, while apparently we cannot conclude that also $(e_1e_2)y' = -e_1(e_2y')$. However, multiplying $(e_1e_2)x = -e_1(e_2x)$ from the left by e_1 we get $e_1((e_1e_2)x) = e_2x$, which can be written as $e_1(e_3x) = -(e_1e_3)x$. Therefore Claim 2 yields $e_1(e_3y') = -(e_1e_3)y'$. Multiplying this from the left by e_1 we arrive at the desired identity $(e_1e_2)y' = -e_1(e_2y')$. Also, $(e_3e_4)y' = -e_3(e_4y')$ holds by Claim 1. We still have to deal with the case where $y' = 0$, i.e., $(e_3e_4)x = e_3(e_4x)$. The usual reasoning now does not work, since we do not have "enough room" to apply Claim 3. Thus, the final conclusion is that there exists a nonzero $y \in A_1$ such that

$$(e_1e_2)y = -e_1(e_2y),\ (e_1e_4)y = -e_1(e_4y),\ (e_2e_4)y = -e_2(e_4y),\ (e_3e_4)y = \pm e_3(e_4y).$$

In view of (b) we may assume without loss of generality that $y^2 = -1$. Let us first consider the case where $(e_3e_4)y = e_3(e_4y)$. We set $f_8 = y$ and $f_i = e_i$, $f_{i+8} = f_if_8$, $i = 1, \dots, 7$. By standard calculations one can now verify that $A \cong \widetilde{\mathbb{S}}$; checking all details is lengthy and tedious, but straigtforward. The other possibility where $(e_3e_4)y = -e_3(e_4y)$ of course leads to $A \cong \mathbb{S}$. □

All lemmas together yield our main result.

Theorem 5.7. *A super-alternative locally complex algebra is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\widetilde{\mathbb{O}}$, $\mathbb{S}$, *or* $\widetilde{\mathbb{S}}$.

Remark 5.8. In the course of the proof we did not use the assumption that (5) holds for all $u, x \in A_1$. Therefore we can replace the super-alternativity assumption by a slightly milder one.

This list reduces to Cayley-Dickson algebras under the additional assumption that there exist alter-scalar elements.

Corollary 5.9. *A super-alternative locally complex algebra containing alter-scalar elements is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, *or* $\mathbb{S}$.

Corollary 5.10. *A super-alternative locally complex algebra which contains alter-scalar elements, but is not alternative, is isomorphic to* $\mathbb{S}$.

Let A be an algebra, and let $x \in A$. The *annihilator* of x is the space $\mathrm{Ann}(x) = \{y \in A \,|\, xy = 0\}$. If $A = \mathbb{A}_n$ is a Cayley-Dickson algebra, then the dimension of $\mathrm{Ann}(x)$ is a multiple of 4 [2, 16]. Moreover, if $A = \mathbb{A}_4 = \mathbb{S}$, then the dimension of $\mathrm{Ann}(x)$ is exactly 4 for every zero divisor x in A [2, Section 12]. The algebras $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbb{S}}$ do not have this property. It is easy to check that $x = f_1 - f_4 \in \widetilde{\mathbb{O}}$ has the 2-dimensional annihilator spanned by $f_2 + f_7$ and $f_3 - f_6$. Further, the dimension of the annihilator of $x = f_3 + f_{12} \in \widetilde{\mathbb{S}}$ is 6; it is spanned by $f_1 + f_{14}$, $f_2 - f_{13}$, $f_4 + f_{11}$, $f_5 + f_{10}$, $f_6 - f_9$, and $f_7 - f_8$. Thus, we have

Corollary 5.11. *Let A be a super-alternative locally complex algebra which is not a division algebra. If the dimension of* $\mathrm{Ann}(x)$ *is* 4 *for every zero divisor in A, then* $A \cong \mathbb{S}$.

One can check that

$$1 \mapsto 1,\ e_1 \mapsto f_1,\ e_2 \mapsto f_2,\ e_3 \mapsto f_3,\ e_4 \mapsto f_{12},\ e_5 \mapsto -f_{13},\ e_6 \mapsto -f_{14},\ e_7 \mapsto -f_{15}$$

defines an embedding of $\widetilde{\mathbb{O}}$ into $\mathbb{S}$. Thus, both $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ can be viewed as subalgebras of $\mathbb{S}$. Chan and Đoković proved that $\mathbb{S}$ has 6-dimensional subalgebras, which, however, are not contained in 8-dimensional subalgebras of $\mathbb{S}$ [6, Corollary 3.6, Theorem 8.1]. Accordingly, $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ do not have 6-dimensional subalgebras. Further, $\mathbb{S}$ does not contain 5-dimensional subalgebras [6, Proposition 4.4]. This does not hold for $\widetilde{\mathbb{S}}$. For example, the linear span of 1, $f_1 + f_{14}$, $f_3 - f_{12}$, $f_6 - f_9$, and $f_7 - f_8$ is a 5-dimensional subalgebra of $\widetilde{\mathbb{S}}$. Combining all these we get our final corollary.

Corollary 5.12. *Let A be a super-alternative locally complex algebra. If A has* 6*-dimensional subalgebras, but does not have* 5*-dimensional subalgebras, then* $A \cong \mathbb{S}$.

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ON LOCALLY COMPLEX ALGEBRAS AND LOW-DIMENSIONAL CAYLEY-DICKSON ALGEBRAS

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Abstract. The paper begins with short proofs of classical theorems by Frobenius and (resp.) Zorn on associative and (resp.) alternative real division algebras. These theorems characterize the first three (resp. four) Cayley-Dickson algebras. Then we introduce and study the class of real unital nonassociative algebras in which the subalgebra generated by any nonscalar element is isomorphic to $\mathbb{C}$. We call them *locally complex algebras*. In particular, we describe all such algebras that have dimension at most 4. Our main motivation, however, for introducing locally complex algebras is that this concept makes it possible for us to extend Frobenius' and Zorn's theorems in a way that it also involves the fifth Cayley-Dickson algebra, the sedenions.

1. Introduction

The real number field $\mathbb{R}$, the complex number field $\mathbb{C}$, and the division agebra of real quaternions $\mathbb{H}$ are classical examples of associative real division algebras. In 1878 Frobenius [10] proved that in the finite dimensional context they are also the only examples. Assuming alternativity instead of associativity, there is another example: $\mathbb{O}$, the division algebra of octonions. It turns out that this is the only additional example. This result is attributed to Zorn [21].

In Section 3 we give short and self-contained proofs of these classical theorems by Frobenius and Zorn. Both proofs are based on the same idea. In fact, the proof of Zorn's theorem is a continuation of the proof of Frobenius' theorem. The proofs are constructive, it appears like $\mathbb{H}$ and $\mathbb{O}$ are met "unintentionally".

Our proofs of Frobenius' and Zorn's theorems were discovered by accident, when examining the class of real unital algebras with the following property: the subalgebra generated by any element different from a scalar multiple of 1 is isomorphic to $\mathbb{C}$. These algebras, which we call *locally complex*, will be first considered in Section 4. In particular, we will classify all locally complex algebras of dimension at most 4.

Unlike real division algebras which exist only in dimensions $1, 2, 4$, and 8 [3, 13], locally complex algebras exist in abundance in any dimension. However, among alternative (and hence also associative) finite dimensional real algebras, the concepts of division algebras and locally complex algebras coincide. Frobenius' and Zorn's theorems can be therefore equivalently stated so that one replaces "division" by "locally complex" in the formulation. This observation paves the way for continuing in the direction of these two theorems.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the first four (real) algebras formed in the Cayley-Dickson process. The next one is the 16-dimensional algebra $\mathbb{S}$ of (real) *sedenions*. It is the first algebra in this process that is neither a division nor an alternative algebra. Although it is therefore somewhat less attractive than its famous predecessors, $\mathbb{S}$ has recently gained a considerable attention. Over the last years it was considered in several papers by algebraists as well as by mathematical physicists [1, 2, 4, 5, 6, 12, 14, 16]. To the best of our knowledge, however, there

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are no results that characterize $\mathbb{S}$ through its abstract algebraic properties. Moreover, one might get an impression when looking at some of these papers that such characterizations are not really expected (for example, see the introduction in [2]). One of the goals of this paper is to show that actually they can be established.

In Section 5 we consider locally complex algebras that are simultaneously superalgebras with the property that all their homogeneous elements satisfy the alternativity conditions (see (1) below). Our main result says that besides the obvious examples, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$, there are exactly two more algebras having these properties, one in dimension 8 and another one in dimension 16. As corollaries we get three characterizations of $\mathbb{S}$: the first one is based on the existence of special elements satisfying a version of the alternativity condition, the second one is based on the properties of zero divisors, and the third one is based on the structure of subalgebras.

Let us remark that among the papers listed above, the one by Calderon and Martin [5] is philosophically the closest one to our paper since it also considers superalgebras. However, the two papers do not seem to have any overlap. On the other hand, in our final results on sedenions we were influenced by the papers [2, 6, 16].

2. Preliminaries

The purpose of this section is to recall some definitions and elementary properties of the notions needed in subsequent sections.

Let A be a nonassociative algebra over a field. In this paper we will be actually interested only in the case where this field is $\mathbb{R}$, although some parts, like the following definitions and comments, make sense in a more general setting. Recall that A is said to be a *division algebra* if for every nonzero $a \in A$, $x \mapsto ax$ and $x \mapsto xa$ are bijective maps from A onto A. If A is finite dimensional, then this is clearly equivalent to the condition that A has no zero divisors. If A is associative, then it is a division algebra if and only if it is unital (i.e., it has a unity 1) and every nonzero element in A has a multiplicative inverse. For general algebras this is not true.

The real *Cayley-Dickson* algebras $\mathbb{A}_n$, $n \geq 0$, are (nonassociative) real algebras with involution $*$, defined recursively as follows: $\mathbb{A}_0 = \mathbb{R}$ with trivial involution $a^* = a$, and $\mathbb{A}_n$ is the vector space $\mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ endowed with multiplication and involution defined by

$$(a,b)(c,d) = (ac - d^*b, da + bc^*),$$

$$(a,b)^* = (a^*, -b).$$

It is easy to see that $\mathbb{A}_n$ is unital (in fact, the unity of $\mathbb{A}_n$ is $(1,0)$ where 1 is the unity of $\mathbb{A}_{n-1}$), $x + x^*$ and $xx^* = x^*x$ are scalar multiplies of 1 for every $x \in \mathbb{A}_n$, and $\dim \mathbb{A}_n = 2^n$. Next, it is clear that $\mathbb{A}_1 = \mathbb{C}$, and one easily notices that $\mathbb{A}_2 = \mathbb{H}$, the *quaternions*. The next algebra in this process is $\mathbb{A}_3 = \mathbb{O}$, the *octonions*. For an excellent survey on octonions we refer the reader to [1]. Let us record here just a few basic properties of $\mathbb{O}$. First of all, $\mathbb{O}$ is an 8-dimensional division algebra. Denoting its basis by $\{1, e_1, \ldots, e_7\}$, the multiplication in $\mathbb{O}$ is determined by the following table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Note that the linear span of $1, e_1, e_2, e_3$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

It is well known that $\mathbb{O}$ is a division algebra which is not associative. However, it is "almost" associative - namely, it is alternative. Recall that an algebra A is said to be *alternative* if

$$x^2y = x(xy) \quad \text{and} \quad yx^2 = (yx)x \tag{1}$$

holds for all $x, y \in A$. Incidentally, Artin's theorem says that this is equivalent to the condition that any two elements generate an associative subalgebra [20, p. 36]. We shall need the identities from (1) in their linearized forms:

$$(xz + zx)y = x(zy) + z(xy), \ y(xz + zx) = (yx)z + (yz)x. \tag{2}$$

Let us also record the so-called middle Moufang identity which, as one easily checks (see, e.g., [20, p. 35]), holds in every alternative algebra:

$$(xy)(zx) = x(yz)x. \tag{3}$$

With regard to the right-hand side of (3) it should be pointed out that alternative algebras are flexible, i.e., $x(yx) = (xy)x$ holds (after all, this follows from Artin's theorem), and therefore there is a convention to write xyx instead of $(xy)x$ or $x(yx)$.

The next algebra obtained by the Cayley-Dickson process is the 16-dimensional algebra $\mathbb{A}_4 = \mathbb{S}$, the *sedenions*. Let $\{1, e_1, \ldots, e_{15}\}$ be a basis of $\mathbb{S}$. This is the multiplication table for $\mathbb{S}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

The sedenions have zero divisors and they are not an alternative algebra. Anyhow, we shall see that they are close enough to alternative division algebras, so that these approximate properties are "almost" characteristic for $\mathbb{S}$. Let us recall the definition of another notion needed for dealing with these properties.

An algebra A is said to be a *superalgebra* if it is $\mathbb{Z}_2$-graded, i.e., there exist linear subspaces A_i, $i \in \mathbb{Z}_2$, such that $A = A_0 \oplus A_1$ and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. We call A_0 an *even* and A_1 an *odd* part of A. Elements in $A_0 \cup A_1$ are said to be *homogeneous*. Note that if A is unital, then $1 \in A_0$.

Cayley-Dickson algebras possess a natural superalgebra structure. Indeed, $A = \mathbb{A}_n$ becomes a superalgebra by defining $A_0 = \mathbb{A}_{n-1} \times 0$ and $A_1 = 0 \times \mathbb{A}_{n-1}$. This simple observation is the concept behind the contents of Section 5.

The algebras $\mathbb{A}_n$, $n \geq 4$, are not alternative, but at least they have certain nonscalar elements that share many properties with elements in alternative algebras: these are scalar multiples of the element $e = (0, 1)$, where 1 is of course the unity of $\mathbb{A}_{n-1}$ (see e.g. [2, Section 5]). Let us point out only one property that is sufficient for our purposes: e satisfies $x^2e = x(xe)$ for all $x \in \mathbb{A}_n$. This can be easily verified. Moreover, this property is "almost" characteristic for e: only elements in the linear span of 1 and e satisfy this identity for every x [8, Lemma 1.2] (the authors are thankful to Alberto Elduque for drawing their attention to this result). Now, let us call an element a in an arbitrary nonassociative algebra A an *alter-scalar* if a is not a scalar and satisfies $x^2a = x(xa)$ holds for all $x \in A$. (A similar, but not exactly the same notion of a strongly alternative element was defined in [17]. There is also a standard notion of an alternative element defined through the condition $a^2x = a(ax)$ for every x, but this is too weak for our goals). What is important for us is that $\mathbb{S}$ contains alter-scalars. With respect to the notation introduced above, these are nonzero scalar multiplies of e_8. Thus, the standard basis of $\mathbb{S}$ has an element that is in some sense "better" than the others. This does not seem to be the case with the preceding Cayley-Dickson algebras.

Next we recall that an algebra A is said to be *quadratic* if it is unital and the elements $1, x, x^2$ are linearly dependent for every $x \in A$. Thus, for every $x \in A$ there exist $t(x), n(x) \in \mathbb{R}$ such that $x^2 - t(x)x + n(x) = 0$. Obviously, $t(x)$ and $n(x)$ are uniquely determined if $x \notin \mathbb{R}$. Setting $t(\lambda) = 2\lambda$ and $n(\lambda) = \lambda^2$ for $\lambda \in \mathbb{R}$, we can then consider t and n as maps from A into $\mathbb{R}$ (the reason for this definition is that in this way t becomes a linear functional, but we shall not need this). We call $t(x)$ and $n(x)$ the *trace* and the *norm* of x, respectively. For some elementary properties of quadratic algebras, a characterization of quadratic alternative algebras, and further references we refer to [9].

From $x^2 - (x+x^*)x + x^*x = 0$ we see that all algebras $\mathbb{A}_n$ are quadratic. Further, every real division algebra A that is algebraic and power-associative (this means that every subalgebra generated by one element is associative) is automatically quadratic. Indeed, if $x \in A$ then there exists a nonzero polynomial $f(X) \in \mathbb{R}[X]$ such that $f(x) = 0$. Writing $f(X)$ as the product of linear and quadratic polynomials in $\mathbb{R}[X]$ it follows that $p(x) = 0$ for some $p(X) \in \mathbb{R}[X]$ of degree 1 or 2. In particular, algebraic alternative (and hence associative) real division algebras are quadratic.

Finally, if A is a real unital algebra, i.e., an algebra over $\mathbb{R}$ with unity 1, then we shall follow a standard convention and identify $\mathbb{R}$ with $\mathbb{R}1$; thus we shall write λ for $\lambda 1$, where $\lambda \in \mathbb{R}$.

3. Frobenius' and Zorn's theorems

Our first lemma is well known. It describes one of the basic properties of quadratic algebras. We give the proof for the sake of completness.

Lemma 3.1. *Let A be a quadratic real algebra. Then $U = \{u \in A \backslash \mathbb{R} \mid u^2 \in \mathbb{R}\} \cup \{0\}$ is a linear subspace of A, $uv + vu \in \mathbb{R}$ for all $u, v \in U$, and $A = \mathbb{R} \oplus U$.*

Proof. Obviously, U is closed under scalar multiplication. We have to show that $u, v \in U$ implies $u + v \in U$. If $u, v, 1$ are linearly dependent, then one easily notices that already u and v are dependent, and the result follows. Thus, let $u, v, 1$ be independent. We have $(u+v)^2 + (u-v)^2 = 2u^2 + 2v^2 \in \mathbb{R}$. On the other hand, as A is quadratic there exist $\lambda, \mu \in \mathbb{R}$ such that $(u+v)^2 - \lambda(u+v) \in \mathbb{R}$ and $(u-v)^2 - \mu(u-v) \in \mathbb{R}$, and hence $\lambda(u+v) + \mu(u-v) \in \mathbb{R}$. However, the independence of $1, u, v$ implies $\lambda + \mu = \lambda - \mu = 0$, so that $\lambda = \mu = 0$. This proves that $u \pm v \in U$. Thus U is indeed a subspace of A. Accordingly, $uv + vu = (u+v)^2 - u^2 - v^2 \in \mathbb{R}$ for all $u, v \in U$. Finally, if $a \in A \setminus \mathbb{R}$, then $a^2 - \nu a \in \mathbb{R}$ for some $\nu \in \mathbb{R}$, and therefore $u = a - \frac{\nu}{2} \in U$; thus, $a = \frac{\nu}{2} + u \in \mathbb{R} \oplus U$. □

Remark 3.2. If A is additionally a division algebra, then every nonzero $u \in U$ can be written as $u = \alpha v$ with $\alpha \in \mathbb{R}$ and $v^2 = -1$. Indeed, since $u^2 \in \mathbb{R}$ and since u^2 cannot be ≥ 0 (otherwise $(u-\alpha)(u+\alpha) = u^2 - \alpha^2$ would be 0 for some $\alpha \in \mathbb{R}$) we have $u^2 = -\alpha^2$ with $0 \neq \alpha \in \mathbb{R}$. Thus, $v = \alpha^{-1}u$ is a desired element.

Note that by $\langle u, v \rangle = -\frac{1}{2}(uv + vu)$ one defines an inner product on U if A is a division algebra. The next lemma therefore deals with nothing but the Gram-Schmidt process. Nevertheless, we give the proof.

Lemma 3.3. *Let A be a quadratic real division algebra, and let U be as in Lemma 3.1. Suppose $e_1, \dots, e_k \in U$ are such that $e_i^2 = -1$ for all $i \leq k$ and $e_ie_j = -e_je_i$ for all $i, j \leq k$, $i \neq j$. If U is not equal to the linear span of $e_1, \dots, e_k$, then there exists $e_{k+1} \in U$ such that $e_{k+1}^2 = -1$ and $e_ie_{k+1} = -e_{k+1}e_i$ for all $i \leq k$.*

Proof. Pick $u \in U$ that is not contained in the linear span of $e_1, \ldots, e_k$, and set $\alpha_i = \frac{1}{2}(ue_i + e_i u) \in \mathbb{R}$ (by Lemma 3.1). Note that $v = u + \alpha_1 e_1 + \ldots + \alpha_k e_k$ satisfies $e_i v = -ve_i$ for all $i \leq k$. Let e_{k+1} be a scalar multiple of v such that $e_{k+1}^2 = -1$ (Remark 3.2). Then e_{k+1} has all desired properties. □

Theorem 3.4. (Frobenius' theorem) *An algebraic associative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. As pointed out at the end of Section 2, A is quadratic. We may assume that $n = \dim A \geq 2$. By Remark 3.2 we can fix $i \in A$ such that $i^2 = -1$. Thus, $A \cong \mathbb{C}$ if $n = 2$. Let $n > 2$. By Lemma 3.3 there is $j \in A$ such that $j^2 = -1$ and $ij = -ji$. Set $k = ij$. Now one immediately checks that $k^2 = -1$, $ki = j = -ik$, $jk = i = -kj$, and i, j, k are linearly independent. Therefore A contains a subalgebra isomorphic to $\mathbb{H}$. It remains to show that n is not > 4. If it was, then by Lemma 3.3 there would exist $e \in A$ such that $e \neq 0$, $ei = -ie$, $ej = -je$, and $ek = -ke$. However, from the first two identities we infer $eij = -iej = ije$; since $ij = k$, this contradicts the third identity. □

In standard graduate algebra textbooks one can find different proofs of Frobenius' theorem. In some of them the advanced theory is used, but there are also such that use only elementary tools, e.g., [11] and [15]. The proof in [11] is actually based on similar ideas than our proof, but it is considerably lengthier. The one in [15] (which is based on [18]) is different, and also short.

We believe that our proof, consisting of four simple steps (Lemma 3.1, Remark 3.2, Lemma 3.3, and the final proof), should be easily understandable to undergraduate students. Some of these steps, especially both lemmas, are of independent interest.

We now switch to the proof of Zorn's theorem. We need a simple lemma:

Lemma 3.5. *Let A be an alternative algebra, and let $e_1, \ldots, e_k \in A$ be such that $e_i e_j \in \{e_1, \ldots, e_k\}$ whenever $i \neq j$. If $w \in A$ is such that $e_i w = -we_i$ for every i, then $(e_i e_j)w = -e_i(e_j w)$ and $w(e_i e_j) = -(we_i)e_j$ whenever $i \neq j$.*

Proof. Just set $x = e_i$, $y = e_j$, and $z = w$ in (2), and the result follows. □

Theorem 3.6. (Zorn's theorem) *An algebraic alternative real division algebra A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. Since a subalgebra generated by two elements is associative, the first part of the proof of Theorem 3.4 remains unchanged in the present context. We may therefore assume that A contains a copy of $\mathbb{H}$ and that $n = \dim A > 4$. Let us just change the notation and write $e_1 = i$, $e_2 = j$, and $e_3 = k$. By Lemma 3.3 there exists $e_4 \in A$ such that $e_4^2 = -1$ and $e_4 e_i = -e_i e_4$ for $i = 1, 2, 3$. Now define $e_5 = e_1 e_4$, $e_6 = e_2 e_4$, $e_7 = e_3 e_4$. Using the alternativity and anticommutativity relations we see that

$$e_5^2 = e_6^2 = e_7^2 = -1,$$

$$e_1 e_5 = -e_5 e_1 = e_2 e_6 = -e_6 e_2 = e_3 e_7 = -e_7 e_3 = -e_4,$$

$$e_4 e_5 = -e_5 e_4 = e_1,\ e_4 e_6 = -e_6 e_4 = e_2,\ e_4 e_7 = -e_7 e_4 = e_3.$$

Further, using (3) we obtain

$$e_5 e_6 = -e_6 e_5 = -e_3,\ e_6 e_7 = -e_7 e_6 = -e_1,\ e_7 e_5 = -e_5 e_7 = -e_2.$$

Finally, use Lemma 3.5 with $k = 3$ and $w = e_4$, and note that the resulting identites yield the rest of the multiplication table.

It is easy to see that $1, e_1, \ldots, e_7$ are linearly independent. Indeed, by taking squares we first see that $\sum_{i=1}^7 \lambda_i e_i$ cannot be a nonzero scalar; if $\sum_{i=1}^7 \lambda_i e_i = 0$, then after multiplying this relation with e_i we get $\lambda_i = 0$. Thus, we have showed that A contains $\mathbb{O}$.

It remains to show that $n = 8$. Suppose $n > 8$. Then, by Lemma 3.3, there exists $f \in A$ such that $f \neq 0$ and $fe_i = -e_i f$, $1 \leq i \leq 7$. Lemma 3.5 tells us that f also satisfies $(e_i e_j)f = -e_i(e_j f)$ and $f(e_i e_j) = -(fe_i)e_j$ for $i \neq j$. Accordingly,

$$e_1(e_2(e_4 f)) = -e_1((e_2 e_4)f) = -e_1(e_6 f) = (e_1 e_6)f = -e_7 f. \tag{4}$$

Note that for $1 \leq i \leq 3$ we have

$$e_i(e_4 f) = -(e_i e_4)f = f(e_i e_4) = -f(e_4 e_i) = (fe_4)e_i = -(e_4 f)e_i.$$

This makes it possible for us to apply Lemma 3.5 for $k = 3$ and $w = e_4 f$. In particular this gives $(e_1 e_2)(e_4 f) = -e_1(e_2(e_4 f))$. Consequently,

$$e_1(e_2(e_4 f)) = -e_3(e_4 f) = (e_3 e_4)f = e_7 f,$$

contradicting (4). □

Remark 3.7. From the first part of the proof we see that if an alternative (not necessarily a division) real algebra A contains a copy of $\mathbb{H}$ and $\dim A > 4$, then it also contains a copy of $\mathbb{O}$.

Classical versions of Frobenius' and Zorn's theorems deal with finite dimensional algebras rather than with (slightly more general) algebraic ones. Our method, however, yields these more general versions for free. But actually we shall need the more general version of Zorn's theorem in Section 5.

We cannot claim that any of the arguments given in this section is entirely original. After finding these proofs we have realized, when searching the literature, that many of these ideas appear in different texts. But to the best of our knowledge nobody has compiled these arguments in the same way that leads to short and direct proofs of theorems by Frobenius and Zorn. Therefore we hope and believe that this section is of some value.

4. Locally complex algebras

As already mentioned, we define a *locally complex algebra* as a real unital algebra A such that every $a \in A \setminus \mathbb{R}$ generates a subalgebra isomorphic to $\mathbb{C}$. A locally complex algebra A is obviously quadratic. We can therefore consider the trace $t(a)$ and the norm $n(a)$ of each $a \in A$.

Lemma 4.1. *The following conditions are equivalent for a real unital algebra A:*

(i) *A is locally complex;*
(ii) *every $0 \neq a \in A$ has a multiplicative inverse lying in $\mathbb{R}a + \mathbb{R}$;*
(iii) *A is quadratic and A has no nontrivial idempotents or square-zero elements;*
(iv) *A is quadratic and $n(a) > 0$ for every $0 \neq a \in A$.*

Moreover, if $2 \leq \dim A = n < \infty$, then (i)-(iv) *are equivalent to*

(v) *A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_i e_j = -e_j e_i$ for all $i \neq j$.*

Proof. It is easy to see that (i)$\Longrightarrow$ (ii) and (ii)$\Longrightarrow$ (iii). Suppose A is quadratic and $n(a) \leq 0$ for some $0 \neq a \in A$. Then $a \notin \mathbb{R}$. Therefore also $b = a - \frac{t(a)}{2} \notin \mathbb{R}$. Note that $b^2 \geq 0$. If $b^2 = 0$, then A has a nontrivial nilpotent. If $b^2 > 0$, i.e., $b^2 = \alpha^2$ for some $0 \neq \alpha \in \mathbb{R}$, then $e = \frac{1}{2}(1 - \alpha^{-1}b)$ is a nontrivial idempotent in A. Thus, (iii)$\Longrightarrow$ (iv). The proof of (iv)$\Longrightarrow$ (ii) is also straightforward. Therefore (ii)-(iv) are equivalent. Now assume (ii)-(iv) and pick $a \in A \setminus \mathbb{R}$. Then $b = a - \frac{t(a)}{2}$ satisfies $b^2 \in \mathbb{R}$. Just as in the argument above we see that b^2 cannot be ≥ 0. Hence $b^2 = -\alpha^2$ for some $\alpha \in \mathbb{R} \setminus \{0\}$, and so $i = \alpha^{-1}b$ satisfies $i^2 = -1$. This yields (i).

Finally, assume $2 \leq \dim A = n < \infty$. The implication (i)-(iv) $\Longrightarrow$ (v) follows from (the proof of) Lemma 3.3. Assuming (v) and writing $a \in A$ as $a = \lambda_0 + \sum_{i=1}^{n-1} \lambda_i e_i$, we see that $a^2 - t(a)a + n(a) = 0$ with $t(a) = 2\lambda_0$ and $n(a) = \sum_{i=0}^{n-1} \lambda_i^2$. Thus, (iv) holds. □

We can now list various examples of locally complex algebras.

Example 4.2. A quadratic real division algebra is locally complex.

Example 4.3. Let J_n be an n-dimensional real vector space, and let $\{1, e_1, \ldots, e_{n-1}\}$ be its basis. Define a multiplication in J_n so that 1 is of course the unity, and the others are multiplied according to $e_i e_j = -\delta_{ij}$. Then J_n is a locally complex algebra and simultaneously a Jordan algebra. Another way of representing J_n is by identifying it with $\mathbb{R} \times \mathbb{R}^{n-1}$, and defining multiplication by $(\lambda, u)(\mu, v) = (\lambda\mu - \langle u, v\rangle, \lambda v + \mu u)$, where $\langle\, .\, ,\, .\, \rangle$ denotes the standard inner product on $\mathbb{R}^{n-1}$.

Example 4.4. A real unital algebra A is said to be *nicely normed* if there exists a linear map $* : A \to A$ such that $a^{**} = a$, $(ab)^* = b^* a^*$ for all $a, b \in A$, and $a + a^* \in \mathbb{R}$, $aa^* = a^* a > 0$ for all $0 \neq a \in A$ (cf. [1, p. 154]). These algebras form an important subclass of locally complex algebras. Namely, every element a in such an algebra A satisfies $a^2 - t(a)a + n(a) = 0$ with $t(a) = a + a^*$ and $n(a) = aa^*$, so that A is indeed locally complex. Note that $U = \{u \in A \setminus \mathbb{R} \,|\, u^2 \in \mathbb{R}\} \cup \{0\} = \{u \in A \,|\, u^* = -u\}$.

In particular, the Cayley-Dickson algebras $\mathbb{A}_n$ are nicely normed, and hence locally complex.

From Lemma 4.1 we can deduce the following characterization of finite dimensional nicely normed algebras.

Corollary 4.5. *let A be a real unital algebra. If $2 \leq \dim A = n < \infty$, then the following conditions are equivalent:*

(i) *A is nicely normed;*
(ii) *A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ such that $e_i^2 = -1$ for all i and $e_i e_j = -e_j e_i \in \operatorname{span}\{e_1, \ldots, e_{n-1}\}$ for all $i \neq j$.*

Proof. Assume (i). By Lemma 4.1 (v) A has a basis $\{1, e_1, \ldots, e_{n-1}\}$ that has all desired properties except that we do not know yet that $e_i e_j \in \operatorname{span}\{e_1, \ldots, e_{n-1}\}$. In view of the observation in Example 4.4 we have $\operatorname{span}\{e_1, \ldots, e_{n-1}\} = U = \{u \in A \,|\, u^* = -u\}$. Therefore, if $i \neq j$, $(e_i e_j)^* = e_j^* e_i^* = e_j e_i = -e_i e_j$, and hence $e_i e_j \in U$. Conversely, if (ii) holds, then we can define $*$ according to $1^* = 1$ and $e_i^* = -e_i$, and one easily checks that this makes A a nicely normed algebra. □

If A is a *commutative* finite dimensional locally complex algebra, then the e_i's from (v) in Lemma 4.1 must satisfy $e_i e_j = 0$ if $i \neq j$. This can be interpreted as follows.

Corollary 4.6. *Let A be a locally complex algebra with $2 \leq \dim A = n < \infty$. Then A is commutative if and only if $A \cong J_n$.*

Let A be an alternative real algebra. If A is an algebraic division algebra, then it is quadratic, and hence , as already mentioned, locally complex. Conversely, if A is locally complex, then by Lemma 4.1 (ii) for every $0 \neq a \in A$ there exist $\lambda, \mu \in \mathbb{R}$ such that $a(\lambda a + \mu) = 1$. Since A is alternative it follows that for every $y \in A$ the equation $ax = y$ has the solution $x = (\lambda a + \mu)y$. Similarly one solves the equation $xa = y$. Therefore A is an algebraic division algebra. Accordingly, Frobenius' and Zorn's theorem can be equivalently stated as follows.

Theorem 4.7. (Frobenius' and Zorn's theorems) *An associative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. An alternative locally complex algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

As already mentioned in the introduction, this version of Frobenius' and Zorn's theorems indicates the direction in which these theorems can be generalized. We shall deal with this in the next section.

In the rest of this section we will classify locally complex algebras up to dimesion 4. Clearly, $\mathbb{R}$ and $\mathbb{C}$ are, up to an isomorphism, the only locally complex algebras of dimension ≤ 2.

We fix some notation. The members of $\mathbb{R}\times\mathbb{R}^2$ will be denoted by $(\lambda, x) = (\lambda, x_1, x_2)$ and the members of $\mathbb{R}\times\mathbb{R}^3$ by $(\lambda, x) = (\lambda, x_1, x_2, x_3)$. For each (ordered) pair $x, y \in \mathbb{R}^2$ we denote by $|x\ y|$ the 2×2 determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. The symbol $x \times y$ stands for the usual vector product (cross product) of $x, y \in \mathbb{R}^3$, while (x, y, z) denotes the scalar triple product $(x, y, z) = \langle x \times y, z\rangle$, $x, y, z \in \mathbb{R}^3$.

Let t, s be nonnegative real numbers. We denote by $A_{t,s}$ the 3-dimensional algebra $A_{t,s} = \mathbb{R}\times\mathbb{R}^2$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda y + \mu x + s|x\ y|e_1),$$

where $e_1 = (1, 0) \in \mathbb{R}^2$. It follows from Lemma 4.1 (v) that $A_{t,s}$ is a locally complex algebra. We will show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t, s) \in [0,\infty)\times[0,\infty)$ and that $A_{t,s}$ and $A_{t',s'}$ are not isomorphic whenever $(t, s) \ne (t', s')$. In short, we have the following classification theorem for 3-dimensional locally complex algebras.

Theorem 4.8. *The map $(t, s) \mapsto A_{t,s}$, $t, s \ge 0$, induces a bijection between $[0,\infty)\times[0,\infty)$ and isomorphism classes of 3-dimensional locally complex algebras.*

Proof. We first show that each 3-dimensional locally complex algebra A is isomorphic to $A_{t,s}$ for some $(t, s) \in [0,\infty)\times[0,\infty)$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R}\times\mathbb{R}^2$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle, \lambda y + \mu x) + |x\ y|(t, z)$$

for some $(t, z) \in \mathbb{R}\times\mathbb{R}^2$. So, we may, and we will assume that A is this algebra. We have two possibilities; either $t \ge 0$, or $t < 0$. Let us consider only the second one; the case when $t \ge 0$ can be handled in a similar, but simpler way. Set $s = \|z\|$. There exists an orthogonal 2×2 matrix Q such that $Qz = -se_1$ and $\det Q = -1$. Observe that $|Qx\ Qy| = (\det Q)|x\ y| = -|x\ y|$ and $\langle Qx, Qy\rangle = \langle x, y\rangle$, $x, y \in \mathbb{R}^2$. We claim that the map $\varphi : A \to A_{|t|,s}$ given by $\varphi(\lambda, x) = (\lambda, Qx)$, $(\lambda, x) \in \mathbb{R}\times\mathbb{R}^2$, is an isomorphism. Clearly, it is linear and bijective. Moreover, we have

$$\begin{aligned}\varphi((\lambda, x)\,(\mu, y)) &= \varphi((\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda y + \mu x + |x\ y|z)) \\ &= (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx - s|x\ y|e_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(\lambda, x)\,\varphi(\mu, y) &= (\lambda, Qx)\,(\mu, Qy) \\ = (\lambda\mu - \langle Qx, Qy\rangle &+ |t|\ |Qx\ Qy|, \lambda Qy + \mu Qx + s|Qx\ Qy|e_1) \\ &= (\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx - s|x\ y|e_1).\end{aligned}$$

Hence, φ is an isomorphism. It remains to show that if $A_{t,s}$ and $A_{t',s'}$ are isomorphic for some $(t, s), (t', s') \in [0,\infty)\times[0,\infty)$, then $(t, s) = (t', s')$.

So, let $\varphi : A_{t,s} \to A_{t',s'}$ be an isomorphism. Then φ is linear and unital. In particular, $\varphi(\lambda, 0) = (\lambda, 0)$ for every $\lambda \in \mathbb{R}$. Furthermore, we have

$$\{(0, x) \in A_{t,s} \,|\, x \in \mathbb{R}^2\} = \{u \in A_{t,s} \,|\, u^2 \in \mathbb{R} \text{ and } u \notin \mathbb{R}\} \cup \{0\}.$$

It follows that

$$\varphi(\lambda, x) = (\lambda, Qx)$$

for some linear map $Q : \mathbb{R}^2 \to \mathbb{R}^2$. From

$$\begin{aligned}(\lambda^2 - \|Qx\|^2, 2\lambda Qx) &= (\lambda, Qx)^2 = (\varphi(\lambda, x))^2 \\ = \varphi((\lambda, x)^2) &= \varphi(\lambda^2 - \|x\|^2, 2\lambda x) = (\lambda^2 - \|x\|^2, 2\lambda Qx)\end{aligned}$$

we get that $\|Qx\|^2 = \|x\|^2$ for every $x \in \mathbb{R}^2$. Thus, Q is orthogonal. The equation

$$\varphi((\lambda, x)\,(\mu, y)) = \varphi(\lambda, x)\,\varphi(\mu, y)$$

can be rewritten as

$$\begin{gathered}(\lambda\mu - \langle x, y\rangle + t|x\ y|, \lambda Qy + \mu Qx + s|x\ y|Qe_1)\\ = (\lambda\mu - \langle x, y\rangle + t'(\det Q)\,|x\ y|, \lambda Qy + \mu Qx + s'(\det Q)\,|x\ y|e_1).\end{gathered}$$

We conclude that $t = t' \det Q$ and $sQe_1 = s'(\det Q)e_1$. Applying the fact that $|\det Q| = 1$ and $\|Qe_1\| = \|e_1\| = 1$ we get $|t| = |t'|$ and $|s| = |s'|$. As all t, t', s, s' are nonnegative, we have $t = t'$ and $s = s'$, as desired. □

It follows directly from Corollary 4.5 that $A_{t,s}$ is nicely normed if and only if $t = 0$. So, the above statement shows that there is a natural bijection between $[0, \infty)$ and isomorphism classes of 3-dimensional nicely normed algebras.

The next result owes a lot to the paper [7] classifying 4-dimensional real quadratic division algebras. Our approach covers a more general class of real algebras. It is self-contained and completely elementary using just simple linear algebra tools.

We identify linear maps on $\mathbb{R}^3$ with 3×3 real matrices. Let M_3 denote the set of all 3×3 real matrices. For $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$ we write $(T, u) \sim (T', u')$ if and only if there exists an orthogonal 3×3 matrix Q such that $T' = (\det Q)QTQ^T$ and $u' = (\det Q)Qu$. It is clear that $\sim$ is an equivalence relation on $M_3 \times \mathbb{R}^3$. The set of equivalence classes will be denoted by $(M_3 \times \mathbb{R}^3)/\sim$.

For $T \in M_3$ and $u \in \mathbb{R}^3$ we denote by $A_{T,u}$ the 4-dimensional algebra $A_{T,u} = \mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y)).$$

As in the 3-dimensional case one can easily verify that $A_{T,u}$ is a locally complex algebra. We will show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$ and that $A_{T,u}$ and $A_{T',u'}$ are isomorphic if and only if $(T, u) \sim (T', u')$. In other words, we will prove the following.

Theorem 4.9. *The map $(T, u) \mapsto A_{T,u}$, $T \in M_3$, $u \in \mathbb{R}^3$, induces a bijection between $(M_3 \times \mathbb{R}^3)/\sim$ and isomorphism classes of 4-dimensional locally complex algebras.*

Proof. We will first show that each 4-dimensional locally complex algebra A is isomorphic to $A_{T,u}$ for some $(T, u) \in M_3 \times \mathbb{R}^3$. It is a straightforward consequence of Lemma 4.1 (v) that A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle, \lambda y + \mu x) + S(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$$

for some linear map $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$. Observe that $S : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ can be decomposed into a direct sum of a linear functional on $\mathbb{R}^3$ and an endomorphism on $\mathbb{R}^3$. Recall that every linear functional on $\mathbb{R}^3$ can be represented in a unique way as an inner product with a fixed vector in $\mathbb{R}^3$. Finally, observe that the coordinates of the vector $(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)$ are up to a permutation and a multiplication by ± 1 the coordinates of the vector product $x \times y$. Thus, A is isomorphic to $\mathbb{R} \times \mathbb{R}^3$ with the multiplication given by

$$(\lambda, x)\,(\mu, y) = (\lambda\mu - \langle x, y\rangle + (x, y, u), \lambda y + \mu x + T(x \times y))$$

for some $u \in \mathbb{R}^3$ and some endomorphism T of $\mathbb{R}^3$. Hence, A is isomorphic to $A_{T,u}$, as desired.

Assume now that $A_{T,u}$ and $A_{T',u'}$ are isomorphic for some $(T, u), (T', u') \in M_3 \times \mathbb{R}^3$. We have to show that $(T, u) \sim (T', u')$.

So, let $\varphi : A_{T,u} \to A_{T',u'}$ be an isomorphism. Exactly in the same way as in the 3-dimensional case we show that

$$\varphi(\lambda, x) = (\lambda, Qx)$$

for some orthogonal 3×3 matrix Q. The equation

$$\varphi((\lambda, x)\,(\mu, y)) = \varphi(\lambda, x)\,\varphi(\mu, y)$$

can be rewritten as

$$(\lambda\mu - \langle x,y\rangle + (x,y,u), \lambda Qy + \mu Qx + QT(x\times y))$$
$$= (\lambda\mu - \langle x,y\rangle + (Qx,Qy,u'), \lambda Qy + \mu Qx + T'(Qx\times Qy)).$$

We conclude that

$$(x,y,u) = (Qx,Qy,u')$$

and

$$QT(x\times y) = T'(Qx\times Qy)$$

for all $x,y\in\mathbb{R}^3$. As Q is orthogonal we have $Q(x\times y) = (\det Q)(Qx\times Qy)$, and consequently,

$$(x,y,u) = (\det Q)\,(x,y,Q^Tu') \quad \text{and} \quad QT(x\times y) = (\det Q)\,T'Q(x\times y), \quad x,y\in\mathbb{R}^3.$$

It follows that $u' = (\det Q)Qu$ and $T' = (\det Q)QTQ^T$, as desired.

Finally, if $(T,u)\sim(T',u')$ for some $T,T'\in M_3$ and $u,u'\in\mathbb{R}^3$ then there exists an orthogonal 3×3 matrix Q such that $T' = (\det Q)QTQ^T$ and $u' = (\det Q)Qu$. It is then straightforward to check that the map $\varphi : A_{T,u}\to A_{T',u'}$ defined by $\varphi(\lambda,x) = (\lambda,Qx)$, $(\lambda,x)\in A_{T,u}$, is an isomorphism. □

It is rather easy to verify that $A_{T,u}$ is nicely normed if and only if $u=0$. We will next show that $A_{T,u}$ is a division algebra if and only if $\langle Tx,x\rangle\neq 0$ for each nonzero $x\in\mathbb{R}^3$ (that is, the quadratic form $q(x)=\langle Tx,x\rangle$ is either positive definite, or negative definite). Indeed, assume first that $A_{T,u}$ is not a division algebra. Then

$$(\lambda\mu - \langle x,y\rangle + (x,y,u), \lambda y + \mu x + T(x\times y)) = 0$$

for some nonzero $(\lambda,x),(\mu,y)\in A_{T,u}$. In particular,

$$T(x\times y) = -\lambda y - \mu x.$$

Set $z = x\times y$. We have $z\neq 0$, since otherwise x and y are linearly dependent and therefore

- either $\lambda=0$ and then $\langle x,y\rangle=0$ and $\mu x=0$ which further yields that $(\lambda,x)=0$ or $(\mu,y)=0$, a contradiction; or
- $\mu=0$ which yields a contradiction in exactly the same way; or
- $\lambda\neq 0$ and $\mu\neq 0$ and then $y=-\mu\lambda^{-1}x$ and $\lambda\mu=\langle x,y\rangle$ yield $0<\lambda^2 = -\langle x,x\rangle\leq 0$, a contradiction.

Hence, $z\neq 0$ and because z is orthogonal to both x and y we have $\langle Tz,z\rangle = 0$.

To prove the other direction we assume that there exists $z\in\mathbb{R}^3$ with $\|z\|=1$ and $\langle Tz,z\rangle=0$. Then $Tz=-tw$ for some real number t and some $w\in\mathbb{R}^3$ with $w\perp z$ and $\|w\|=1$. There is a unique $v\in\mathbb{R}^3$ such that $z=w\times v$ and $v\perp w$. Set $s=-(w,v,u)$. Then $(0,w)$ and $(t,v-sw)$ are nonzero elements of $A_{T,u}$ whose product is equal to zero. Hence, $A_{T,u}$ is not a division algebra, as desired.

Following Dieterich's idea [7] we will now disscuss a geometric interpretation of the classification of 4-dimensional locally complex algebras. Let us start with a simple observation concerning 3×3 skew-symmetric matrices. If $x,y\in\mathbb{R}^3$ are any two vectors such that $x\times y=(c_1,c_2,c_3)$, then

$$R = \begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{bmatrix} = xy^T - yx^T,$$

where x and y are represented as 3×1 matrices. If Q is any orthogonal matrix, then $QRQ^T = (Qx)(Qy)^T - (Qy)(Qx)^T$. As $Qx\times Qy = (\det Q)\,Q(x\times y)$, we have

$$Q\begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{bmatrix}Q^T = \begin{bmatrix} 0 & d_3 & -d_2 \\ -d_3 & 0 & d_1 \\ d_2 & -d_1 & 0 \end{bmatrix},$$

where

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = (\det Q)\, Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

If we choose $Q \in SO(3)$ such that

$$\begin{bmatrix} 0 \\ 0 \\ \sqrt{c_1^2 + c_2^2 + c_3^2} \end{bmatrix} = Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

then

$$QRQ^T = \begin{bmatrix} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $d = \sqrt{c_1^2 + c_2^2 + c_3^2}$. In particular, $d = \|R\|$.

Any 3×3 matrix T can be uniquely decomposed into its symmetric and skew-symmetric part, $T = P + R$, $P = (1/2)(T + T^T)$, $R = (1/2)(T - T^T)$. If $T' = (\det Q)QTQ^T$ and $T' = P' + R'$ with P' symmetric and R' skew-symmetric, then $P' = (\det Q)QPQ^T$ and $R' = (\det Q)QRQ^T$. We will say that $A_{T,u}$ is of rank 3,2,1,0, respectively, if the symmetric part P of T is of rank 3,2,1,0, respectively. By the previous remark, two isomorphic algebras $A_{T,u}$ have the same rank.

Let us start with algebras $A_{T,u}$ of rank 3. We have two possibilities: either all eigenvalues of $P = T + T^T$ have the same sign, or P has both positive and negative eigenvalues. In the first case we will say that $A_{T,u}$ is an ellipsoid locally complex algebra of dimension 4, while in the second case we call $A_{T,u}$ a hyperboloid locally complex algebra of dimension 4. As we are interested in isomorphism classes we can use the fact that $A_{T,u}$ is isomorphic to $A_{-T,u}$ to restrict our attention to the case when all the eigenvalues of P are positive (the ellipsoid case) or to the case when two eigenvalues of P are positive and one is negative (the hyperboloid case). Once we have done this restriction two algebras $A_{T,u}$ and $A_{T',u'}$ of the above types are isomorphic if and only if $T' = QTQ^T$ and $u' = Qu$ for some $Q \in SO(3)$.

To consider isomorphism classes of hyperboloid locally complex algebras of dimension 4 (a 4-dimensional locally complex algebra is hyperboloid if it is isomorphic to some hyperboloid algebra $A_{T,u}$) we set $\tau = \{\delta \in \mathbb{R}^3 \,|\, \delta_1 \geq \delta_2 > 0 > \delta_3\}$ and $\kappa = \tau \times \mathbb{R}^3 \times \mathbb{R}^3$. The elements of κ will be called configurations. Each configuration consists of a hyperboloid $H_\delta = \{x \in \mathbb{R}^3 \,|\, \langle \Delta_\delta x, x\rangle = 1\}$ (a hyperboloid in principal axis form) and a pair of points. Here, Δ_δ is the diagonal matrix with the diagonal entries: $\delta_1, \delta_2, \delta_3$. The symmetry group of the hyperboloid H_δ is defined to be $G_\delta = \{Q \in SO(3) \,|\, Q\Delta_\delta Q^T = \Delta_\delta\}$ (the requirement that $\det Q = 1$ tells that we allow only symmetries that preserve the orientation). Note that this symmetry group consists of 4 elements whenever $\delta_1 > \delta_2$. Namely, in this case the symmetry group consists of the identity and all diagonal matrices with two eigenvalues -1 and one eigenvalue 1. The symmetry group is infinite if and only if the hyperboloid H_δ is circular, that is, $\delta_1 = \delta_2$. Two configurations (δ, u, c) and (δ', u', c') are said to be equivalent, $(\delta, u, c) \equiv (\delta', u', c')$, if and only if their hyperboloids coincide and their pairs of points lie in the same orbit under the operation of the symmetry group of the hyperboloid, that is, if and only if $\delta = \delta'$ and $(u', c') = (Qu, Qc)$ for some $Q \in G_\delta$. We denote by $\kappa/\equiv$ the set of equivalence classes of κ. We have a natural bijection between $\kappa/\equiv$ and the set of equivalence classes of hyperboloid locally complex algebras of dimension 4. Indeed, the bijection is induced by the map

$$(\delta, u, c) \mapsto A_{\Delta_\delta + R_c, u}$$

where

$$\Delta_\delta + R_c = \begin{bmatrix} \delta_1 & c_3 & -c_2 \\ -c_3 & \delta_2 & c_1 \\ c_2 & -c_1 & \delta_3 \end{bmatrix}.$$

Clearly, $A_{\Delta_\delta+R_c,u}$ is a hyperboloid locally complex algebra. We have to show that each hyperboloid algebra $A_{T,v}$ is isomorphic to some $A_{\Delta_\delta+R_c,u}$ and that $A_{\Delta_\delta+R_c,u}$ and $A_{\Delta_{\delta'}+R_{c'},u'}$ are isomorphic if and only if $(\delta,u,c) \equiv (\delta',u',c')$. The second statement is trivial. To verify the first one we write $T = P + R$ with P symmetric with two positive eigenvalues and R skew-symmetric. Then there exists $Q \in SO(3)$ such that $QPQ^T = \Delta_\delta$ for some $\delta \in \tau$. We have $QRQ^T = R_c$ for some $c \in \mathbb{R}^3$. Set $u = Qv$ to complete the proof.

In a similar fashion we can consider isomorphism classes of ellipsoid locally complex algebras of dimension 4. Note that a locally complex algebra $A_{T,u}$ is a division algebra if and only if it is an ellipsoid algebra. As above we can consider configurations which consist of an ellipsoid in principal axis form and a pair of points. To each such configuration there corresponds a 4-dimensional real division algebra and this correspondence induces a bijection between the equivalence classes of configurations (the equivalence being defined via the symmetry group of the ellipsoid) and the isomorphism classes of 4-dimensional real quadratic division algebras. We omit the details that can be found in [7]. It is clear that locally complex algebras of rank 2 are either elliptic cylinder algebras or hyperbolic cylinder algebras. We leave the details to the reader. In the same way one can classify also isomorphism classes of locally complex algebras of rank 1. Let us conclude with the detailed disscussion on 4-dimensional locally complex algebras of rank 0. By e_3 we denote $e_3 = (0,0,1) \in \mathbb{R}^3$. We define an equivalence relation on the set $[0,\infty) \times \mathbb{R}^3$ as follows: $(d,u),(d',u') \in [0,\infty) \times \mathbb{R}^3$ are said to be equivalent, $(d,u) \equiv (d',u')$, if either

- $d = d' = 0$ and $\|u\| = \|u'\|$; or
- $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3\rangle = \langle u', e_3\rangle$.

Note that the equivalence class of $(d,u) \in [0,\infty) \times \mathbb{R}^3$ with $d > 0$ contains infinitely many elements if u and e_3 are linearly independent, and is a singleton when u is a scalar multiple of e_3. There is a natural bijection between the isomorphism classes of 4-dimensional locally complex algebras of rank 0 and the set $([0,\infty) \times \mathbb{R}^3)/\equiv$. The bijection is induced by the map from $[0,\infty) \times \mathbb{R}^3$ which maps the pair (d,u), $d \geq 0$, $u \in \mathbb{R}^3$, into $A_{T_d,u}$ with

$$T_d = \begin{bmatrix} 0 & d & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $A_{T_d,u}$ is a locally complex algebra of rank 0 and one can easily verify that each 4-dimensional locally complex algebra of rank 0 is isomorphic to some $A_{T_d,u}$. It remains to show that $A_{T_d,u}$ and $A_{T_{d'},u'}$ are isomorphic if and only if $(d,u) \equiv (d',u')$. So, assume that $A_{T_d,u}$ and $A_{T_{d'},u'}$ are isomorphic for some $(d,u),(d',u') \in [0,\infty) \times \mathbb{R}^3$. Then there exists an orthogonal matrix Q such that $T_{d'} = (\det Q)QT_dQ^T$ and $u' = (\det Q)Qu$. In particular, $d' = \|T_{d'}\| = \|T_d\| = d$ and $\|u'\| = \|u\|$. If $d = 0$, then $d' = 0$, and hence, $(d,u) \equiv (d',u')$ in this special case. Therefore we may assume that $d = d' > 0$. From $T_{d'} = (\det Q)QT_dQ^T$ we conclude that $Qe_3 = (\det Q)e_3$. Consequently,

$$\langle u', e_3\rangle = \langle (\det Q)Qu, (\det Q)Qe_3\rangle = \langle u, e_3\rangle.$$

To prove the converse we assume that $(d,u) \equiv (d',u')$. We have one of the two possibilities and we will consider just the second one. So, assume that $d = d' > 0$, $\|u\| = \|u'\|$, and $\langle u, e_3\rangle = \langle u', e_3\rangle$. Then there exists an orthogonal matrix Q such

that $Qe_3 = e_3$ and $Qu = u'$. The orthogonal complement of e_3 and u is one-dimensional (if e_3 and u are linearly independent) or two-dimensional (if e_3 and u are linearly dependent). We have a freedom to choose the action of Q on the orthogonal complement of e_3 and u (of course, up to the requirement that Q is an orthogonal matrix). In particular, we can choose Q in such a way that $\det Q = 1$. It follows that $T_{d'} = QT_dQ^T$ and $u' = Qu$, as desired.

5. SUPER-ALTERNATIVE LOCALLY COMPLEX ALGEBRAS

Let us call an algebra A a *super-alternative algebra* if it is $\mathbb{Z}_2$-graded, $A = A_0 \oplus A_1$, and the alternativity conditions (1) hold for all its homogeneous elements. Equivalently,

$$u^2x = u(ux),\ xu^2 = (xu)u \quad \text{for all } u \in A_i,\ i \in \mathbb{Z}_2,\ x \in A, \tag{5}$$

or, in the linearized form,

$$\begin{aligned} &(uv + vu)x = u(vx) + v(ux), \\ &x(uv + vu) = (xu)v + (xv)u \quad \text{for all } u, v \in A_i,\ i \in \mathbb{Z}_2\ ,\ x \in A. \end{aligned} \tag{6}$$

The notion of a super-alternative algebra should not be confused with the notion of an *alternative superalgebra*. The latter is defined through the alternativity of the Grassmann envelope of A. It turns out that nontrivial examples of alternative superalgebras exist only very exceptionally: prime alternative superalgebras of characteristic different from 2 and 3 are either associative or their odd part is zero [19]. As we shall see, super-alternative algebras are more easy to find.

Throughout this section A *will be a super-alternative locally complex algebra.* Our goal is to to classify all such algebras A. Obvious examples are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, as we can always take the trivial $\mathbb{Z}_2$-grading (the odd part is 0). Further, one can check by a straigtforward calculation that if $\mathbb{A}_{n-1}$ is an alternative algebra, then every $u \in (\mathbb{A}_{n-1} \times 0) \cup (0 \times \mathbb{A}_{n-1})$ satisfies (5) for every $x \in \mathbb{A}_n$. Therefore, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{S}$ are super-alternative algebras with respect to the natural $\mathbb{Z}_2$-grading mentioned in Section 2. Of course, the important information for us in this context is that $\mathbb{S}$ is also a super-alternative locally complex algebra. As we shall see, besides $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and $\mathbb{S}$ only two more algebras must be added to the complete list of such algebras.

We continue by recording several simple but useful observations. First, the following special case of (6) will be often used:

(a) If $u, v \in A_i$, $i \in \mathbb{Z}_2$, are such that $uv + vu = 0$, then $u(vx) = -v(ux)$ and $(xu)v = -(xv)u$ for all $x \in A$.

If $v \in A_1$, then $v^2 \in A_0$; on the other hand, $v^2 = \lambda v + \mu$ for some $\lambda, \mu \in \mathbb{R}$. Since $v \notin A_0$, we must have $\lambda = 0$ and hence $v^2 = \mu \in \mathbb{R}$. Since A is locally complex, it follows that $\mu < 0$ if $v \neq 0$. Thus, we have

(b) If $0 \neq v \in A_1$, then there is $\alpha \in \mathbb{R}$ such that $(\alpha v)^2 = -1$.

Let $u \in A_0$ and $v \in A_1$ be such that $u^2 = v^2 = -1$. Using Lemma 3.1 we have $uv + vu \in \mathbb{R} \cap A_1 = 0$. Therefore $v(uv) = -v(vu) = -v^2u = u$. Next, $(uv)v = uv^2 = -u$. Similarly we see that $(uv)u = -u(uv) = v$. Finally, using (a) we get $(uv)(uv) = -(uv)(vu) = v((uv)u) = v^2 = -1$. We have proved:

(c) If $u \in A_0$ and $v \in A_1$ are such that $u^2 = v^2 = -1$, then $uv = -vu$, $v(uv) = -(uv)v = u$, $(uv)u = -u(uv) = v$, and $(uv)^2 = -1$.

Let u be a homogeneous element and suppose that $ux = 0$ for some $x \in A$. If $u \neq 0$, then by multiplying this identity from the left by $u - t(u)$ it follows from (5) that $n(u)x = 0$, and hence $x = 0$. Similarly, $xu = 0$ implies $x = 0$ if $u \neq 0$. Thus:

(d) Homogeneous elements are not zero divisors.

It is clear that our conditions on A imply that A_0 is a locally complex alternative algebra. Theorem 4.7 therefore tells us that A_0 is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. If $A_1 = 0$, then we get the desired conclusion that $A = A_0$ is one of the algebras from the expected list. Without loss of generality we may therefore assume that $A_1 \neq 0$. Given $0 \neq u \in A_1$, it follows from (d) that $x \mapsto ux$ is an injective linear map from A_0 into A_1; the same rule defines an injective linear map from A_1 into A_0. We may therefore conclude that

(e) $\dim A_0 = \dim A_1$.

In particular we now know that a super-alternative locally complex algebra must be finite dimensional. Moreover, its dimension can be only 1, 2, 4, 8, or 16.

We shall now consider separately each of the four possibilities concerning A_0.

Lemma 5.1. *If* $A_0 \cong \mathbb{R}$, *then* $A \cong \mathbb{C}$.

Proof. By (b) there is $i \in A_1$ with $i^2 = -1$, and hence $A \cong \mathbb{C}$ by (e). ☐

Lemma 5.2. *If* $A_0 \cong \mathbb{C}$, *then* $A \cong \mathbb{H}$.

Proof. We have $A_0 = \mathbb{R} \oplus \mathbb{R}i$ with $i^2 = -1$. By (b) we may pick $j \in A_1$ such that $j^2 = -1$. Setting $k = ij \in A_1$ it follows from (c) that A contains a copy of $\mathbb{H}$. However, in view of (e) we actually have $A \cong \mathbb{H}$. ☐

Let us now introduce another (an unexpected one for us) example of a super-alternative locally complex algebra. Let $\widetilde{\mathbb{O}}$ be the 8-dimensional algebra with basis $\{1, f_1, \ldots, f_7\}$ and multiplication table

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	f_7	$-f_6$
f_2	$-f_3$	-1	f_1	f_6	$-f_7$	$-f_4$	f_5
f_3	f_2	$-f_1$	-1	f_7	f_6	$-f_5$	$-f_4$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3
f_5	f_4	f_7	$-f_6$	$-f_1$	-1	f_3	$-f_2$
f_6	$-f_7$	f_4	f_5	$-f_2$	$-f_3$	-1	f_1
f_7	f_6	$-f_5$	f_4	$-f_3$	f_2	$-f_1$	-1

Lemma 5.3. $\widetilde{\mathbb{O}}$ *is a super-alternative locally complex algebra with zero divisors and without alter-scalar elements (and hence* $\widetilde{\mathbb{O}} \ncong \mathbb{O}$*).*

Proof. The fact that $\widetilde{\mathbb{O}}$ is locally complex follows from Lemma 4.1 (v). Let $\widetilde{\mathbb{O}}_0$ be the linear span of $1, f_1, f_2, f_3$, and let $\widetilde{\mathbb{O}}_1$ be the linear span of f_4, f_5, f_6, f_7. Then $\widetilde{\mathbb{O}}$ becomes a superalgebra with the even part $\widetilde{\mathbb{O}}_0 \cong \mathbb{H}$. From the way we shall arrive at $\widetilde{\mathbb{O}}$ in the next proof it is not really surprising that $\widetilde{\mathbb{O}}$ is super-alternative. But we used Mathematica for the actual checking that this is indeed true. Note that $(f_1 - f_4)(f_3 - f_6) = 0$, so that $\widetilde{\mathbb{O}}$ has zero divisors. Let $a \in \widetilde{\mathbb{O}}$ be such that $x^2a = x(xa)$ for all $x \in \widetilde{\mathbb{O}}$. From $(f_i + f_j)^2a = (f_i + f_j)((f_i + f_j)a)$, together with $f_i(f_ia) = f_j(f_ja) = -a$, it follows that $f_i(f_ja) + f_j(f_ia) = 0$ whenever $i \neq j$. Writing $a = \lambda_0 + \sum_{k=1}^{7} \lambda_k f_k$ we thus have

$$\sum_{k=1}^{7} \lambda_k \Big(f_i(f_jf_k) + f_j(f_if_k) \Big) = 0 \quad \text{whenever } i \neq j. \tag{7}$$

Chosing $i = 1$ and $j = 4$ it follows that $\lambda_2 = \lambda_3 = \lambda_6 = \lambda_7 = 0$. Chosing, for example, $i = 2$ and $j = 7$ we further get $\lambda_1 = \lambda_4 = 0$, and chosing $i = 3$ and $j = 4$ finally leads to $\lambda_5 = 0$. Therefore $a = \lambda_0$ is a scalar. ☐

Lemma 5.4. *If* $A_0 \cong \mathbb{H}$, *then* $A \cong \mathbb{O}$ *or* $A \cong \widetilde{\mathbb{O}}$.

Proof. Let $\{1, i, j, k\}$ be a basis of A_0 where these elements have the usual meaning. Pick $f \in A_1$ with $f^2 = -1$. Then f anticommutes with i, j, k by (c). It is clear that $\{f, if, jf, kf\}$ is a basis of A_1. We claim that all elements in this basis pairwise anticommute. It is easy to see that f anticommutes with each of if, jf, kf. Using (a) repeatedly we obtain $(if)(jf) = -(i(jf))f = (j(if))f = -(jf)(if)$. Other identities can be checked analogously.

Since $i(jf) \in A_1$, we have

$$i(jf) = \lambda_1 f + \lambda_2 if + \lambda_3 jf + \lambda_4 kf \tag{8}$$

for some $\lambda_i \in \mathbb{R}$. From (a) we infer that $(i(jf))f = -(if)(jf)$. Similarly, using (a) and (c) we get

$$f(i(jf)) = -f((jf)i) = (jf)(fi) = -(jf)(if) = (if)(jf).$$

The last two identities show that $i(jf)$ anticommutes with f. Consequently, anticommuting (8) with f it follows that $\lambda_1 = 0$. A similar arguing shows that $i(jf)$ anticommutes with both if and jf, which leads to $\lambda_2 = \lambda_3 = 0$. Note that (c) implies that the squares of both kf and $i(jf)$ are equal -1. But then $\lambda_4^2 = 1$, i.e., $\lambda_4 = 1$ or $\lambda_4 = -1$. If $\lambda_4 = 1$, i.e., $i(jf) = kf$, then we set $f_1 = i$, $f_2 = j$, $f_3 = k$, $f_4 = f$, $f_5 = if$, $f_6 = jf$, and $f_7 = kf$. Using the information we have, it is now just a matter of a routine calculation to verify that $A \cong \widetilde{\mathbb{O}}$. Since we know that $\mathbb{O}$ is a super-alternative locally complex algebra, the other possibility $\lambda_4 = -1$ can lead only to $A \cong \mathbb{O}$. □

The 16-dimensional analogue of $\widetilde{\mathbb{O}}$ is the algebra which we denote by $\widetilde{\mathbb{S}}$ and define as follows: if $\{1, f_1, \ldots, f_{15}\}$ is its basis, then the multiplication table is

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	$-f_7$	f_6	f_9	$-f_8$	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}
f_2	$-f_3$	-1	f_1	f_6	f_7	$-f_4$	$-f_5$	f_{10}	f_{11}	$-f_8$	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$
f_3	f_2	$-f_1$	-1	f_7	$-f_6$	f_5	$-f_4$	f_{11}	$-f_{10}$	f_9	$-f_8$	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$
f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3	f_{12}	f_{13}	f_{14}	$-f_{15}$	$-f_8$	$-f_9$	$-f_{10}$	f_{11}
f_5	f_4	$-f_7$	f_6	$-f_1$	-1	$-f_3$	f_2	f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	$-f_8$	f_{11}	f_{10}
f_6	f_7	f_4	$-f_5$	$-f_2$	f_3	-1	$-f_1$	f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	$-f_8$	$-f_9$
f_7	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1	f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	$-f_8$
f_8	$-f_9$	$-f_{10}$	$-f_{11}$	$-f_{12}$	$-f_{13}$	$-f_{14}$	$-f_{15}$	-1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_9	f_8	$-f_{11}$	f_{10}	$-f_{13}$	f_{12}	$-f_{15}$	f_{14}	$-f_1$	-1	$-f_3$	f_2	$-f_5$	f_4	$-f_7$	f_6
f_{10}	f_{11}	f_8	$-f_9$	$-f_{14}$	f_{15}	f_{12}	$-f_{13}$	$-f_2$	f_3	-1	$-f_1$	$-f_6$	f_7	f_4	$-f_5$
f_{11}	$-f_{10}$	f_9	f_8	f_{15}	f_{14}	$-f_{13}$	$-f_{12}$	$-f_3$	$-f_2$	f_1	-1	f_7	f_6	$-f_5$	$-f_4$
f_{12}	f_{13}	f_{14}	$-f_{15}$	f_8	$-f_9$	$-f_{10}$	f_{11}	$-f_4$	f_5	f_6	$-f_7$	-1	$-f_1$	$-f_2$	f_3
f_{13}	$-f_{12}$	$-f_{15}$	$-f_{14}$	f_9	f_8	f_{11}	f_{10}	$-f_5$	$-f_4$	$-f_7$	$-f_6$	f_1	-1	f_3	f_2
f_{14}	f_{15}	$-f_{12}$	f_{13}	f_{10}	$-f_{11}$	f_8	$-f_9$	$-f_6$	f_7	$-f_4$	f_5	f_2	$-f_3$	-1	$-f_1$
f_{15}	$-f_{14}$	f_{13}	f_{12}	$-f_{11}$	$-f_{10}$	f_9	f_8	$-f_7$	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1

The proof of the next lemma is similar to that of Lemma 5.3. Therefore we omit details.

Lemma 5.5. *$\widetilde{\mathbb{S}}$ is a super-alternative locally complex algebra without alter-scalar elements (and hence $\widetilde{\mathbb{S}} \ncong \mathbb{S}$).*

The final lemma is similar to Lemma 5.4, but the proof is somewhat more complicated. One of the problems that we have to face in this proof is that we do not have a complete freedom in the selection of an element playing the role of f from the proof of Lemma 5.4. While f was an arbitrary element in A_1 with square -1, now we shall have to find a special one.

Lemma 5.6. *If $A_0 \cong \mathbb{O}$, then $A \cong \mathbb{S}$ or $A \cong \widetilde{\mathbb{S}}$.*

Proof. Let $\{1, e_1, \ldots, e_7\}$ be a basis of A_0 whose multiplication table is given in Section 2. We begin with three claims needed for future reference.

CLAIM 1: Let $i, j \in \{1, 2, \ldots, 7\}$, $i \neq j$. If $p \in A_1$, then $q = p + (e_i e_j)(e_i(e_j p))$ satisfies $(e_i e_j)q = -e_i(e_j q)$.

Indeed, by (5) we have $(e_ie_j)q = (e_ie_j)p - e_i(e_jp)$, while using (a) and (5) we get

$$\begin{aligned} e_i(e_jq) &= e_i(e_jp) + e_i(e_j((e_ie_j)(e_i(e_jp)))) = e_i(e_jp) - e_i((e_ie_j)(e_j(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_i(e_j(e_i(e_jp)))) = e_i(e_jp) - (e_ie_j)(e_j(e_i(e_i(e_jp)))) \\ &= e_i(e_jp) + (e_ie_j)(e_j(e_jp)) = e_i(e_jp) - (e_ie_j)p, \end{aligned}$$

so that $(e_ie_j)q = -e_i(e_jq)$.

Claim 2: Let $i, j, k \in \{1, 2, \dots, 7\}$ be such that e_i, e_j, e_ie_j, e_k are linearly independent, and let $s \in A_1$ be such that $(e_ie_j)s = -e_i(e_js)$. Then $t = s + (e_ie_k)(e_i(e_ks))$ also satisfies $(e_ie_j)t = -e_i(e_jt)$.

(Let us add that (a) implies $t = s + (e_ke_i)(e_k(e_is))$, and that $(e_ie_j)z = -e_i(e_jz)$ is equivalent to $(e_je_i)z = -e_j(e_iz)$; the order of indices is thus irrelevant.)

Indeed, by now already familiar arguing we have

$$\begin{aligned} (e_ie_j)t &= (e_ie_j)s + (e_ie_j)((e_ie_k)(e_i(e_ks))) = (e_ie_j)s - (e_ie_k)((e_ie_j)(e_i(e_ks))) \\ &= (e_ie_j)s + (e_ie_k)(e_i((e_ie_j)(e_ks))) = (e_ie_j)s - (e_ie_k)(e_i(e_k((e_ie_j)s))) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_i(e_k(e_i(e_js))))\big) = -\big(e_i(e_js) + (e_ie_k)(e_k(e_i(e_i(e_js))))\big) \\ &= -\big(e_i(e_js) - (e_ie_k)(e_k(e_js))\big) = -\big(e_i(e_js) + e_i(e_i((e_ie_k)(e_k(e_js))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_i(e_k(e_js))))\big) = -\big(e_i(e_js) + e_i((e_ie_k)(e_i(e_j(e_ks))))\big) \\ &= -\big(e_i(e_js) - e_i((e_ie_k)(e_j(e_i(e_ks))))\big) = -\big(e_i(e_js) + e_i(e_j((e_ie_k)(e_i(e_ks))))\big) \\ &= -e_i(e_jt). \end{aligned}$$

Claim 3: Let $i, j, k \in \{1, 2, \dots, 7\}$, $i \neq j$, and let $\epsilon \in \mathbb{R}$ and $w \in A_1$ be such that $(e_ie_j)w = \epsilon e_i(e_jw)$. Set $u = e_kw$. If $k \in \{i, j\}$, then $(e_ie_j)u = \epsilon e_i(e_ju)$, and if $k \notin \{i, j\}$, then $(e_ie_j)u = -\epsilon e_i(e_ju)$.

If $k \in \{i, j\}$, then we may assume $k = j$ without loss of generality. We have

$$(e_ie_j)(u) = (e_ie_j)(e_jw) = -e_j((e_ie_j)w) = -\epsilon e_j(e_i(e_jw)) = \epsilon e_i(e_ju).$$

If $k \notin \{i, j\}$, then we have

$$\begin{aligned} (e_ie_j)(u) &= (e_ie_j)(e_kw) = -e_k((e_ie_j)w) \\ &= -\epsilon e_k(e_i(e_jw)) = \epsilon e_i(e_k(e_jw)) = -\epsilon e_i(e_ju). \end{aligned}$$

After establishing these auxiliary claims, we now begin the actual proof by picking a nonzero $u \in A_1$. As mentioned above, an arbitrary chosen u may not be the right choice, so we have to "remedy" it. Let $v' = u + (e_1e_2)(e_1(e_2u)) \in A_1$. By Claim 1, v' satisfies $(e_1e_2)v' = -e_1(e_2v')$. If $v' = 0$, then we have $(e_1e_2)u = e_1(e_2u)$. But then $v'' = e_3u$ satisfies $(e_1e_2)v'' = -e_1(e_2v'')$ by Claim 3. Thus, in any case there is a nonzero $v \in A_1$ such that

$$(e_1e_2)v = -e_1(e_2v).$$

Now consider $w' = v + (e_1e_4)(e_1(e_4v))$. By Claim 1 we have $(e_1e_4)w' = -e_1(e_4w')$, and by Claim 2 we have $(e_1e_2)w' = -e_1(e_2w')$. If $w' = 0$, then $(e_1e_4)v = e_1(e_4v)$. But then $w'' = e_2v$ satisfies $(e_1e_2)w'' = -e_1(e_2w'')$ and $(e_1e_4)w'' = -e_1(e_4w'')$. Thus, there exists a nonzero $w \in A_1$ satisfying

$$(e_1e_2)w = -e_1(e_2w), \ (e_1e_4)w = -e_1(e_4w).$$

We now repeat the same procedure with respect to e_2 and e_4. That is, we introduce $x' = w + (e_2e_4)(e_2(e_4w))$, and apply Claims 1 and 2 to conclude that $(e_1e_2)x' = -e_1(e_2x')$, $(e_1e_4)x' = -e_1(e_4x')$, and $(e_2e_4)x' = -e_2(e_4x')$. If $x' = 0$, then $(e_2e_4)w = e_2(e_4w)$, and therefore Claim 3 tells us that $(e_1e_2)x'' = -e_1(e_2x'')$, $(e_1e_4)x'' = -e_1(e_4x'')$, and $(e_2e_4)x'' = -e_2(e_4x'')$, where $x'' = e_1w$. In any case we have found a a nonzero $x \in A_1$ satisfying

$$(e_1e_2)x = -e_1(e_2x), \ (e_1e_4)x = -e_1(e_4x), \ (e_2e_4)x = -e_2(e_4x).$$

Considering $y' = x + (e_3e_4)(e_3(e_4x))$ we see from Claim 2 that $(e_1e_4)y' = -e_1(e_4y')$ and $(e_2e_4)y' = -e_2(e_4y')$, while apparently we cannot conclude that also $(e_1e_2)y' = -e_1(e_2y')$. However, multiplying $(e_1e_2)x = -e_1(e_2x)$ from the left by e_1 we get $e_1((e_1e_2)x) = e_2x$, which can be written as $e_1(e_3x) = -(e_1e_3)x$. Therefore Claim 2 yields $e_1(e_3y') = -(e_1e_3)y'$. Multiplying this from the left by e_1 we arrive at the desired identity $(e_1e_2)y' = -e_1(e_2y')$. Also, $(e_3e_4)y' = -e_3(e_4y')$ holds by Claim 1. We still have to deal with the case where $y' = 0$, i.e., $(e_3e_4)x = e_3(e_4x)$. The usual reasoning now does not work, since we do not have "enough room" to apply Claim 3. Thus, the final conclusion is that there exists a nonzero $y \in A_1$ such that

$$(e_1e_2)y = -e_1(e_2y),\ (e_1e_4)y = -e_1(e_4y),\ (e_2e_4)y = -e_2(e_4y),\ (e_3e_4)y = \pm e_3(e_4y).$$

In view of (b) we may assume without loss of generality that $y^2 = -1$. Let us first consider the case where $(e_3e_4)y = e_3(e_4y)$. We set $f_8 = y$ and $f_i = e_i$, $f_{i+8} = f_if_8$, $i = 1, \dots, 7$. By standard calculations one can now verify that $A \cong \widetilde{\mathbb{S}}$; checking all details is lengthy and tedious, but straigtforward. The other possibility where $(e_3e_4)y = -e_3(e_4y)$ of course leads to $A \cong \mathbb{S}$. □

All lemmas together yield our main result.

Theorem 5.7. *A super-alternative locally complex algebra is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\widetilde{\mathbb{O}}$, $\mathbb{S}$, *or* $\widetilde{\mathbb{S}}$.

Remark 5.8. In the course of the proof we did not use the assumption that (5) holds for all $u, x \in A_1$. Therefore we can replace the super-alternativity assumption by a slightly milder one.

This list reduces to Cayley-Dickson algebras under the additional assumption that there exist alter-scalar elements.

Corollary 5.9. *A super-alternative locally complex algebra containing alter-scalar elements is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, *or* $\mathbb{S}$.

Corollary 5.10. *A super-alternative locally complex algebra which contains alter-scalar elements, but is not alternative, is isomorphic to* $\mathbb{S}$.

Let A be an algebra, and let $x \in A$. The *annihilator* of x is the space $\operatorname{Ann}(x) = \{y \in A \,|\, xy = 0\}$. If $A = \mathbb{A}_n$ is a Cayley-Dickson algebra, then the dimension of $\operatorname{Ann}(x)$ is a multiple of 4 [2, 16]. Moreover, if $A = \mathbb{A}_4 = \mathbb{S}$, then the dimension of $\operatorname{Ann}(x)$ is exactly 4 for every zero divisor x in A [2, Section 12]. The algebras $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbb{S}}$ do not have this property. It is easy to check that $x = f_1 - f_4 \in \widetilde{\mathbb{O}}$ has the 2-dimensional annihilator spanned by $f_2 + f_7$ and $f_3 - f_6$. Further, the dimension of the annihilator of $x = f_3 + f_{12} \in \widetilde{\mathbb{S}}$ is 6; it is spanned by $f_1 + f_{14}$, $f_2 - f_{13}$, $f_4 + f_{11}$, $f_5 + f_{10}$, $f_6 - f_9$, and $f_7 - f_8$. Thus, we have

Corollary 5.11. *Let A be a super-alternative locally complex algebra which is not a division algebra. If the dimension of* $\operatorname{Ann}(x)$ *is* 4 *for every zero divisor in A, then* $A \cong \mathbb{S}$.

One can check that

$$1 \mapsto 1,\ e_1 \mapsto f_1,\ e_2 \mapsto f_2,\ e_3 \mapsto f_3,\ e_4 \mapsto f_{12},\ e_5 \mapsto -f_{13},\ e_6 \mapsto -f_{14},\ e_7 \mapsto -f_{15}$$

defines an embedding of $\widetilde{\mathbb{O}}$ into $\mathbb{S}$. Thus, both $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ can be viewed as subalgebras of $\mathbb{S}$. Chan and Đoković proved that $\mathbb{S}$ has 6-dimensional subalgebras, which, however, are not contained in 8-dimensional subalgebras of $\mathbb{S}$ [6, Corollary 3.6, Theorem 8.1]. Accordingly, $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ do not have 6-dimensional subalgebras. Further, $\mathbb{S}$ does not contain 5-dimensional subalgebras [6, Proposition 4.4]. This does not hold for $\widetilde{\mathbb{S}}$. For example, the linear span of 1, $f_1 + f_{14}$, $f_3 - f_{12}$, $f_6 - f_9$, and $f_7 - f_8$ is a 5-dimensional subalgebra of $\widetilde{\mathbb{S}}$. Combining all these we get our final corollary.

Corollary 5.12. *Let A be a super-alternative locally complex algebra. If A has 6-dimensional subalgebras, but does not have 5-dimensional subalgebras, then $A \cong \mathbb{S}$.*

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